On Simpson's rule and fractional Brownian motion with H = 1/10

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Abstract

We consider stochastic integration with respect to fractional Brownian motion (fBm) with $H < 1/2$. The integral is constructed as the limit, where it exists, of a sequence of Riemann sums. A theorem by Gradinaru, Nourdin, Russo & Vallois (2005) holds that a sequence of Simpson's rule Riemann sums converges in probability for a sufficiently smooth integrand f and when the stochastic process is fBm with $H > 1/10$. For the case $H = 1/10$, we prove that the sequence of sums converges in distribution. Consequently, we have an Itô-like formula for the resulting stochastic integral. The convergence in distribution follows from a Malliavin calculus theorem that first appeared in Nourdin and Nualart (2010).

1 Introduction

Let $B = \{B_t^H, t \geq 0\}$ be a fractional Brownian motion (fBm), that is, B is a centered Gaussian process with covariance given by

$$\mathbb{E}\left[B_s B_t\right] := R(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \tag{1}$$

for $s, t \geq 0$, where $H \in (0,1)$ is the Hurst parameter. For a smooth function $f : \mathbb{R} \to \mathbb{R}$, we take the 'Simpson's rule' Riemann sum with uniform partition,

$$S_n^S(t) := \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{6}\left(f'(B_{\frac{j}{n}}) + 4f'\left((B_{\frac{j}{n}} + B_{\frac{j+1}{n}})/2\right) + f'(B_{\frac{j+1}{n}})\right)\left(B_{\frac{j+1}{n}} - B_{\frac{j}{n}}\right).$$

It can be shown (see [3], or Section 3.1) that this sequence of sums converges in probability when B is fBm with $H > 1/10$, but in general it does not converge in probability when $H \leq 1/10$. In this paper, we consider the particular case of $H = 1/10$, and show that $S_n^S(t)$ does converge weakly to a random variable. More precisely, Theorem 3.3 shows that, conditioned on the path $\{B_s, s \leq t\}$,

$$S_n^S(t) \xrightarrow{\mathcal{L}} f(B_t) - f(0) + \frac{\beta}{2880}\int_0^t f^{(5)}(B_s)dW_s, \tag{2}$$

where W_t is a standard Brownian motion, independent of B, and β is a constant defined in Theorem 3.3. This result allows us to write the change-of-variable formula

$$f(B_t) \stackrel{\mathcal{L}}{=} f(0) + \int_0^t f'(B_s)d^S B_s - \frac{\beta}{2880}\int_0^t f^{(5)}(B_s)dW_s, \tag{3}$$

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where the differential $d^S B_s$ denotes the limit of the Simpson's rule sum.

Conditional convergence in distribution follows from a central limit theorem given in Section 2 (Theorem 2.3). This is a new version of a theorem that first appeared in Nourdin and Nualart (2010) [6]. This theorem uses Malliavin calculus, and applies to a random vector with components in the form of Malliavin divergence integrals. After proving Theorem 2.3, the main task in proving (3) is to verify the conditions of Theorem 2.3, which are relatively long and technical.

1.1 Background.

Assuming a uniform partition, the classical Stratonovich stochastic integral is defined as

$$\int_0^t f'(B_s)d^\circ B_s = \lim_{n\to\infty} S_n^T(t) := \lim_{n\to\infty} \sum_{j=0}^{\lfloor nt\rfloor -1} \frac{1}{2}\left(f'(B_{\frac{j}{n}}) + f'(B_{\frac{j+1}{n}})\right)\left(B_{\frac{j+1}{n}} - B_{\frac{j}{n}}\right), \tag{4}$$

provided that limit exists. It has been shown that this limit exists in probability when B is a fBm with $H > 1/6$, but does not, in general converge in probability for $H \leq 1/6$ (see [2, 3, 8], also Section 3.1). Subsequently, it was proved in [8] that for $H = 1/6$, (4) does converge in law to a random variable that includes a Wiener-Itô integral, that is, as $n \to \infty$

$$S_n^T(t) \xrightarrow{\mathcal{L}} f(B_t) - f(0) + \gamma \int_0^t f^{(3)}(B_s)dW_s,$$

where γ is a known constant and W is a standard Brownian motion, independent of B. Hence, there is the change-of-variable formula

$$f(B_t) \stackrel{\mathcal{L}}{=} f(0) + \int_0^t f'(B_s)d^\circ B_s - \gamma \int_0^t f^{(3)}(B_s)\; dW_s. \tag{5}$$

The reader will recognize that (4) is the Riemann sum corresponding to the 'Trapezoidal rule' of basic calculus. It is certainly possible to generalize to other types of Riemann sums. The 'Midpoint' sum,

$$\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor} f'(B_{\frac{2j-1}{n}})\left(B_{\frac{2j}{n}} - B_{\frac{2j-2}{n}}\right),$$

can be shown to converge in probability for fBm with $H > 1/4$ (see [11]). The end point case $H = 1/4$ was considered in papers by Burdzy and Swanson [1], and Nourdin and Réveillac [7]. These papers proved the change-of-variable formula

$$f(B_t) \stackrel{\mathcal{L}}{=} f(0) + \int_0^t f'(B_s)d^\star B_s + \theta \int_0^t f''(B_s)dW_s, \tag{6}$$

where θ is a constant, W is a scaled Brownian motion, independent of B, and the notation $d^\star B_s$ denotes the integral arising from the midpoint sum.

1.2 Extensions.

Following the results (5) and (6), the present authors also wrote papers on the cases $H = 1/4$ and $H = 1/6$ [4, 5]. These papers contained alternate proofs of (6) and (5), using Malliavin calculus and a version of Theorem 2.3. An interesting difference in the present paper, is that the sum $S_n^S(t)$ converges conditionally to a random variable that is actually the sum of two, independent Gaussian random variables. In the cases considered in [4, 5], there was only a single random term. In those prior papers, we also showed that the results could be extended to other Gaussian processes sufficiently

similar to fBm, for example, bifractional Brownian motion with $HK = 1/6$ in the case of (5). It was also shown that the Midpoint and Trapezoidal Riemann sums converge as functions in the Skorohod space $\mathbf{D}[0,\infty)$, by proving that the sums converge in the sense of finite-dimensional distributions. We expect that similar extensions could be applied to the present Theorem 3.3, but we have not pursued this in the present paper.

We also expect that the techniques of this paper could be applied to the 'Milne's rule' sum for the case $H = 1/14$, see Proposition 3.1.

The organization of this paper is as follows: in Section 2, we give a brief description of the Malliavin calculus definitions and identities that will be used. We also discuss properties of fBm, and prove the central limit theorem which will be applied for the main result. In Section 3, after a brief introduction we state and prove the main result, which is Theorem 3.3. Finally, Section 4 contains proofs of three of the longer lemmas from Section 3.

2 Notation and Theory

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and N be a Gaussian random variable with mean zero and variance σ^2. We say that f satisfies *moderate growth conditions* if there exist constants A, B, and $\alpha < 2$ such that $|f(x)| \le Ae^{B|x|^\alpha}$. Note that this implies $\mathbb{E}\left[|f(N)|^p\right] < \infty$ for all $p \ge 1$. We use the symbol $\mathbf{1}_{[0,t]}$ to denote the indicator function for a real interval $[0,t]$. The symbol C denotes a generic positive constant, which may vary from line to line. In general, the value of C will depend on and the growth conditions of a test function f and the properties of a stochastic process B.

2.1 Elements of Malliavin Calculus.

Following is a brief description of some identities that will be used in the paper. The reader may refer to [9] for detailed coverage of this topic. Let $Z = \{Z(h), h \in \mathcal{H}\}$ be an *isonormal Gaussian process* on a probability space $(\Omega, \mathcal{F}, P)$, and indexed by a real separable Hilbert space $\mathcal{H}$. That is, Z is a family of Gaussian random variables such that $\mathbb{E}[Z(h)] = 0$ and $\mathbb{E}\left[Z(h)Z(g)\right] = \langle h, g\rangle_{\mathcal{H}}$ for all $h, g \in \mathcal{H}$. We will assume that $\mathcal{F}$ is the $\sigma-$algebra generated by Z.

For integers $q \ge 1$, let $\mathcal{H}^{\otimes q}$ denote the q^{th} tensor product of $\mathcal{H}$, and $\mathcal{H}^{\odot q}$ denote the subspace of symmetric elements of $\mathcal{H}^{\otimes q}$. We will also use the notation $\bigotimes_{i=1}^r h_i$ to denote an arbitrary tensor product, with the convention that $\bigotimes_{i=1}^0$ is the empty set.

Let $\{e_n, n \ge 1\}$ be a complete orthormal system in $\mathcal{H}$. For functions $f, g \in \mathcal{H}^{\odot q}$ and $p \in \{0, \ldots, q\}$, we define the p^{th}-order contraction of f and g as that element of $\mathcal{H}^{\otimes 2(q-p)}$ given by

$$f \otimes_p g = \sum_{i_1,\ldots,i_p=1}^{\infty} \left\langle f, e_{i_1} \otimes \cdots \otimes e_{i_p}\right\rangle_{\mathcal{H}^{\otimes p}} \otimes \left\langle g, e_{i_1} \otimes \cdots \otimes e_{i_p}\right\rangle_{\mathcal{H}^{\otimes p}} \tag{7}$$

where $f \otimes_0 g = f \otimes g$ and $f \otimes_q g = \langle f, g\rangle_{\mathcal{H}^{\otimes q}}$. While f, g are symmetric, the contraction $f \otimes_q g$ may not be. We denote its symmetrization by $f \widetilde{\otimes}_q g$.

Let $\mathcal{H}_q$ be the q^{th} Wiener chaos of Z, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(Z(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where $H_q(x)$ is the q^{th} Hermite polynomial, defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}},$$

and we follow the convention of Hermite polynomials with unity as a leading coefficient. For $q \ge 1$, it is known that the map

$$I_q(h^{\otimes q}) = H_q(Z(h)) \tag{8}$$

provides a linear isometry between $\mathcal{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\|\cdot\|_{\mathcal{H}^{\otimes q}}$) and $\mathcal{H}_q$, where $I_q(\cdot)$ is the generalized Wiener-Itô multiple stochastic integral. By convention, $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$. It follows from (8) and the properties of the Hermite polynomials that for $f \in \mathcal{H}^{\odot p}$, $g \in \mathcal{H}^{\odot q}$ we have

$$\mathbb{E}\left[I_p(f)I_q(g)\right] = \begin{cases} p!\left\langle f, g\right\rangle_{\mathcal{H}^{\otimes p}} & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form $F = g(Z(\phi_1), \ldots, Z(\phi_n))$, where $n \geq 1$; $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support, and $\phi_i \in \mathcal{H}$. The Malliavin derivative of F with respect to Z is the element of $L^2(\Omega; \mathcal{H})$ defined as

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(Z(\phi_1), \ldots, Z(\phi_n))\phi_i.$$

By iteration, for any integer $q > 1$ we can define the q^{th} derivative $D^q F$, which is an element of $L^2(\Omega; \mathcal{H}^{\odot q})$.

We let $\mathbb{D}^{q,2}$ denote the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{q,2}}$ defined as

$$\|F\|^2_{\mathbb{D}^{q,2}} = \mathbb{E}\left[F^2\right] + \sum_{i=1}^{q} \mathbb{E}\left[\|D^i F\|^2_{\mathcal{H}^{\otimes i}}\right].$$

More generally, for any Hilbert space V, let $\mathbb{D}^{k,p}(V)$ denote the corresponding Sobolev space of $V-$valued random variables.

We denote by δ the Skorohod integral, which is defined as the adjoint of the operator D. A random element $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of δ, Dom δ, if and only if,

$$|\mathbb{E}\left[\langle DF, u\rangle_{\mathcal{H}}\right]| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant which depends only on u. If $u \in$ Dom δ, then the random variable $\delta(u) \in L^2(\Omega)$ is defined for all $F \in \mathbb{D}^{1,2}$ by the duality relationship,

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathcal{H}}\right].$$

This is sometimes called the Malliavin integration by parts formula. We iteratively define the multiple Skorohod integral for $q \geq 1$ as $\delta(\delta^{q-1}(u))$, with $\delta^0(u) = u$. For this definition we have,

$$\mathbb{E}\left[F\delta^q(u)\right] = \mathbb{E}\left[\langle D^q F, u\rangle_{\mathcal{H}^{\otimes q}}\right], \tag{10}$$

where $u \in$ Dom δ^q and $F \in \mathbb{D}^{q,2}$. The adjoint operator δ^q is an integral in the sense that for a (non-random) $h \in \mathcal{H}^{\odot q}$, we have $\delta^q(h) = I_q(h)$.

The following results will be used extensively in this paper. The reader may refer to [6] and [9] for proofs and details.

Lemma 2.1. *Let $q \geq 1$ be an integer, and $r, j, k > 0$ be integers.*

(a) Assume $F \in \mathbb{D}^{q,2}$, u is a symmetric element of Dom δ^q*, and $\left\langle D^r F, \delta^j(u)\right\rangle_{\mathcal{H}^{\otimes r}} \in L^2(\Omega; \mathcal{H}^{\otimes q-r-j})$ for all $0 \leq r + j \leq q$. Then $\langle D^r F, u\rangle_{\mathcal{H}^{\otimes r}} \in$* Dom δ^r *and*

$$F\delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}\left(\langle D^r F, u\rangle_{\mathcal{H}^{\otimes r}}\right).$$

(b) Suppose that u is a symmetric element of $\mathbb{D}^{j+k,2}(\mathcal{H}^{\otimes j})$. Then we have,

$$D^k\delta^j(u) = \sum_{i=0}^{j\wedge k} i!\binom{k}{i}\binom{j}{i}\delta^{j-i}\left(D^{k-i}u\right).$$

(c) Meyer Inequality: *Let $p > 1$ and integers $k \geq q \geq 1$. Then for any $u \in \mathbb{D}^{k,p}(\mathcal{H}^{\otimes q})$,*

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p}\,\|u\|_{\mathbb{D}^{k,p}}\,(\mathcal{H}^{\otimes q}),$$

where $c_{k,p}$ is a constant.

(d) Let $u \in \mathcal{H}^{\odot p}$ and $v \in \mathcal{H}^{\odot q}$. Then

$$\delta^p(u)\delta^q(v) = \sum_{z=0}^{p\wedge q} z!\binom{p}{z}\binom{q}{z}\delta^{p+q-2z}(u \otimes_z v),$$

where $\otimes_z$ is the contraction operator defined in (7).

2.2 A convergence theorem.

Definition 2.2. Assume F_n is a sequence of $d-$dimensional random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, and F is a $d-$dimensional random variable defined on $(\Omega, \mathcal{G}, P)$, where $\mathcal{F} \subset \mathcal{G}$. We say that F_n *converges stably* to F as $n \to \infty$, if, for any continuous and bounded function $f : \mathbb{R}^d \to \mathbb{R}$ and $\mathbb{R}$-valued, $\mathcal{F}-$measurable random variable M, we have

$$\lim_{n\to\infty} \mathbb{E}\left(f(F_n)M\right) = \mathbb{E}\left(f(F)M\right).$$

The first version of the following central limit theorem appeared in [6]. In [4], we extended this to a multi-dimensional version, where the sequence was a vector of d components all in the same Wiener chaos. For our present paper, we need a slight modification. In this version, we lay out conditions for stable convergence of a sequence of vectors, where the vector components are not necessarily in the same Wiener chaos.

Theorem 2.3. *Let $d \geq 1$ be an integer, and $q_1, \ldots, q_d$ be positive integers with $q^* = \max\{q_1, \ldots, q_d\}$. Suppose that F_n is a sequence of random variables in $\mathbb{R}^d$ of the form $F_n = \left(\delta^{q_1}(u_n^1), \ldots, \delta^{q_d}(u_n^d)\right)$, where each u_n^i is a $\mathbb{R}-$valued symmetric function in $\mathbb{D}^{2q^*,2q_i}(\mathcal{H}^{\otimes q_i})$. Suppose that the sequence F_n is bounded in $L^1(\Omega)$ and that:*

(a) $\left\langle u_n^j, \bigotimes_{\ell=1}^m (D^{a_\ell}F_n^{j_\ell}) \otimes h\right\rangle_{\mathcal{H}^{\otimes q}}$ converges to zero in $L^1(\Omega)$ for all integers $1 \leq j, j_\ell \leq d$, all integers $1 \leq a_1, \ldots, a_m, r \leq q_j - 1$ such that $a_1 + \cdots + a_m + r = q_j$; and all $h \in \mathcal{H}^{\otimes r}$.

(b) For each $1 \leq i, j \leq d$, $\left\langle u_n^i, D^{q_i}F_n^i\right\rangle_{\mathcal{H}^{\otimes q_i}}$ converges in $L^1(\Omega)$ to a nonnegative random variable s_i^2, and for $i \neq j$, $\left\langle u_n^i, D^{q_i}F_n^j\right\rangle_{\mathcal{H}^{\otimes q_i}}$ converges to zero in $L^1(\Omega)$.

Then F_n converges stably to a random vector in $\mathbb{R}^d$, whose components each have independent Gaussian law $\mathcal{N}(0, s_i^2)$ given Z.

Proof. This proof mostly follows that given in [4], except in that case there was only a single value of q. We use the conditional characteristic function. Given any $h_1, \ldots h_m \in \mathcal{H}$, we want to show that the sequence

$$\xi_n = \left(F_n^1, \ldots, F_n^d, Z(h_1), \ldots, Z(h_m)\right)$$

converges in distribution to a vector $\left(F_\infty^1, \dots F_\infty^d, Z(h_1), \dots, Z(h_m)\right)$, where, for any vector $\lambda \in \mathbb{R}^d$, F_∞ satisfies

$$\mathbb{E}\left(e^{i\lambda^T F_\infty} | Z(h_1), \dots, Z(h_m)\right) = \exp\left(-\frac{1}{2}\lambda^T S \lambda\right), \tag{11}$$

where S is the diagonal $d \times d$ matrix with entries s_i^2.

Since F_n is bounded in $L^1(\Omega)$, the sequence ξ_n is tight in the sense that for any $\varepsilon > 0$, there is a $K > 0$ such that $P\left(F_n \in [-K, K]^d\right) > 1 - \varepsilon$, which follows from Chebyshev inequality. Dropping to a subsequence if necessary, we may assume that ξ_n converges in distribution to a limit $\left(F_\infty^1, \dots F_\infty^d, Z(h_1), \dots Z(h_m)\right)$. Let $Y := g\left(Z(h_1), \dots, Z(h_m)\right)$, where $g \in \mathcal{C}_b^\infty(\mathbb{R}^m)$, and consider $\phi_n(\lambda) = \phi(\lambda, \xi_n) := \mathbb{E}\left(e^{i\lambda^T F_n} Y\right)$ for $\lambda \in \mathbb{R}^d$. The convergence in law of ξ_n implies that for each $1 \le j \le d$:

$$\lim_{n\to\infty} \frac{\partial \phi_n}{\partial \lambda_j} = \lim_{n\to\infty} i\mathbb{E}\left(F_n^j e^{i\lambda^T F_n} Y\right) = i\mathbb{E}\left(F_\infty^j e^{i\lambda^T F_\infty} Y\right), \tag{12}$$

where convergence in distribution follows from a truncation argument applied to F_n^j.

On the other hand, using the duality property of the Skorohod integral and the Malliavin derivative:

$$\begin{aligned}
\frac{\partial \phi_n}{\partial \lambda_j} &= i\mathbb{E}\left(\delta^{q_j}(u_n^j) e^{i\lambda^T F_n} Y\right) = i\mathbb{E}\left(\left\langle u_n^j, D^{q_j}\left(e^{i\lambda^T F_n} Y\right)\right\rangle_{\mathfrak{H}^{\otimes q_j}}\right) \\
&= i\sum_{a=0}^{q_j} \binom{q_j}{a} \mathbb{E}\left(\left\langle u_n^j, D^a\left(e^{i\lambda^T F_n}\right) \tilde{\otimes} D^{q_j - a} Y\right\rangle_{\mathfrak{H}^{\otimes q_j}}\right) \\
&= i\left\{\mathbb{E}\left\langle u_n^j, Y D^{q_j} e^{i\lambda^T F_n}\right\rangle_{\mathcal{H}^{\otimes q_j}} + \sum_{a=0}^{q_j-1} \binom{q_j}{a} \mathbb{E}\left\langle u_n^j, D^a e^{i\lambda^T F_n} \tilde{\otimes} D^{q_j-a} Y\right\rangle_{\mathcal{H}^{\otimes q_j}}\right\}
\end{aligned} \tag{13}$$

By condition (a), we have that $\left\langle u_n^j, D^a e^{i\lambda^T F_n} \tilde{\otimes} D^{q_j-a} Y\right\rangle_{\mathcal{H}^{\otimes q_j}}$ converges to zero in $L^1(\Omega)$ when $a < q_j$, so the sum term vanishes as $n \to \infty$, and this leaves

$$\begin{aligned}
\lim_{n\to\infty} i\mathbb{E}\left\langle u_n^j, Y D^q e^{i\lambda^T F_n}\right\rangle_{\mathcal{H}^{\otimes q_j}} &= \lim_{n\to\infty} i\sum_{k=1}^d \mathbb{E}\left(i\lambda_k e^{i\lambda^T F_n} \langle u_n^j, Y D^{q_j} F_n^k\rangle_{\mathcal{H}^{\otimes q_j}}\right) \\
&= -\mathbb{E}\left(\lambda_j e^{i\lambda^T F_\infty} s_j^2 Y\right)
\end{aligned}$$

because the lower-order derivatives in $D^{q_j} e^{i\lambda^T F_n}$ also vanish by condition (a), and cross terms ($j \neq k$) terms vanish by condition (b). Combining this with (12), we obtain:

$$i\mathbb{E}\left(F_\infty^j e^{i\lambda\cdot F_\infty} Y\right) = -\lambda_j \mathbb{E}\left(e^{i\lambda\cdot F_\infty} s_j^2 Y\right).$$

This leads to the PDE system:

$$\frac{\partial}{\partial \lambda_j}\mathbb{E}\left(e^{i\lambda^T F_\infty} | Z(h_1), \dots, Z(h_m)\right) = -\lambda_j s_j^2 \mathbb{E}\left(e^{i\lambda^T F_\infty} | Z(h_1), \dots, Z(h_m)\right)$$

which has unique solution (11). □

Remark 2.4. It suffices to impose condition (a) for $h \in \mathcal{S}_0$, where $\mathcal{S}_0$ is a total subset of $\mathcal{H}^{\otimes r}$.

Remark 2.5. Suppose F_n is the vector sequence (F_n, G_n), where $F_n = \delta^p(u_n)$ and $G_n = \delta^q(v_n)$. Then to satisfy Theorem 2.3, F_n and G_n must be bounded in $L^1(\Omega)$, and the following terms must tend to zero in $L^1(\Omega)$:

1. $\langle u_n, h\rangle_{\mathcal{H}^{\otimes p}}$ and $\langle v_n, g\rangle_{\mathcal{H}^{\otimes q}}$, for arbitrary $h \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$, respectively.

2. $\left\langle u_n, \bigotimes_{i=1}^s D^{a_i}F_n \bigotimes_{i=s+1}^r D^{a_i}G_n \otimes h\right\rangle_{\mathcal{H}^{\otimes p}}$, where $0 \le a_i < p$, $a_1 + \cdots + a_r < p$, and $h \in \mathcal{H}^{\otimes p-(a_1+\cdots+a_r)}$; and $\left\langle u_n, \bigotimes_{i=1}^s D^{a_i}F_n \bigotimes_{i=s+1}^r D^{a_i}G_n\right\rangle_{\mathcal{H}^{\otimes p}}$, where $0 \le a_i < p$ and $a_1+\cdots+a_r = p$.

3. $\left\langle v_n, \bigotimes_{i=1}^s D^{a_i}F_n \bigotimes_{i=s+1}^r D^{a_i}G_n \otimes h\right\rangle_{\mathcal{H}^{\otimes q}}$, where $0 \le a_i < q$, $a_1 + \cdots + a_r < q$, and $h \in \mathcal{H}^{\otimes q-(a_1+\cdots+a_r)}$; and $\left\langle v_n, \bigotimes_{i=1}^s D^{a_i}F_n \bigotimes_{i=s+1}^r D^{a_i}G_n\right\rangle_{\mathcal{H}^{\otimes q}}$, where $0 \le a_i < q$ and $a_1+\cdots+a_r = q$.

4. $\langle u_n, D^pG_n\rangle_{\mathcal{H}^{\otimes p}}$ and $\langle v_n, D^qF_n\rangle_{\mathcal{H}^{\otimes q}}$.

Then for condition (b), the following two terms must converge in $L^1(\Omega)$ to nonnegative random variables: $\langle u_n, D^pF_n\rangle_{\mathcal{H}^{\otimes p}}$ and $\langle v_n, D^qG_n\rangle_{\mathcal{H}^{\otimes q}}$.

2.3 Fractional Brownian motion.

For some $T > 0$, let $B = \{B_t^H, 0 \le t \le T\}$ be a fractional Brownian motion with Hurst parameter H. That is, B is a centered Gaussian process with covariance $R(s,t)$ given in (1). Let $\mathcal{E}$ denote the set of $\mathbb{R}$-valued step functions on $[0,T]$. We then let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product

$$\left\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]}\right\rangle_{\mathfrak{H}} = R(s,t).$$

The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space spanned by B. In this way, $\{B(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process as in Section 2.1.

For an integer $n \ge 2$, we consider a uniform partition of $[0,\infty)$ given by $\{j/n, j \ge 1\}$. Define the following notation:

- $\Delta B_{\frac{j}{n}} = B_{\frac{j+1}{n}} - B_{\frac{j}{n}}$, and $\widetilde{B}_{\frac{j}{n}} = \frac{1}{2}\left(B_{\frac{j}{n}} + B_{\frac{j+1}{n}}\right)$
- $\partial_{\frac{j}{n}} = \mathbf{1}_{[\frac{j}{n},\frac{j+1}{n}]}$, $\varepsilon_t = \mathbf{1}_{[0,t]}$, and $\widetilde{\varepsilon}_{\frac{j}{n}} = \frac{1}{2}\left(\mathbf{1}_{[0,\frac{j}{n}]} + \mathbf{1}_{[0,\frac{j+1}{n}]}\right) = \varepsilon_{\frac{j}{n}} + \frac{1}{2}\partial_{\frac{j}{n}}$.

Assume $H < 1/2$. The following fBm properties follow from (1).

(B.1) $\mathbb{E}\left[\Delta B_{\frac{j}{n}}^2\right] = \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}} = n^{-2H}$.

(B.2) $\mathbb{E}\left[\Delta B_{\frac{j}{n}}\Delta B_{\frac{j+1}{n}}\right] = \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{j+1}{n}}\right\rangle_{\mathfrak{H}} = (2^{2H}-2)/2n^{2H}$.

(B.3) If $|k-j| \ge 2$, $\left|\mathbb{E}\left[\Delta B_{\frac{j}{n}}\Delta B_{\frac{k}{n}}\right]\right| = \left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right| \le Cn^{-2H}|j-k|^{2H-2}$, where the constant C does not depend on j.

(B.4) For each $j \ge 0$, $\sup_{t\in[0,T]}\left|\mathbb{E}\left[\Delta B_{\frac{j}{n}} B_t\right]\right| \le 2n^{-2H}$.

(B.5) For any $t \in [0,T]$ and integer $j \ge 1$, $\left|\mathbb{E}\left[\Delta B_{\frac{j}{n}} B_t\right]\right| = \left|\left\langle \partial_{\frac{j}{n}}, \varepsilon_t\right\rangle_{\mathfrak{H}}\right| \le Cn^{-2H}\left(j^{2H-1} + |j-nt|^{2H-1}\right)$.
In particular, if $|k-j| \ge 2$, $\left|\mathbb{E}\left[\Delta B_{\frac{j}{n}}\widetilde{B}_{\frac{k}{n}}\right]\right| = \left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right| \le n^{-2H}\left(j^{2H-1} + |j-k|^{2H-1}\right)$.

As a result of properties (B.1) - (B.5), we have the following technical results.

Lemma 2.6. *Let $H < 1/2$ and $0 < t \le T$, and let $n \ge 2$ be an integer. Then*

(a) For fixed $0 \le s \le T$ and integer $r \ge 1$,

$$\sum_{j=0}^{\lfloor nt\rfloor - 1}\left|\left\langle \partial_{\frac{j}{n}}, \varepsilon_s\right\rangle_{\mathfrak{H}}^r\right| \le Cn^{-2(r-1)H}.$$

(b) For integer $r \geq 1$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}^{r} \right| \leq C n^{-2(r-1)H}.$$

(c) For integers $r \geq 1$ and $0 \leq k \leq \lfloor nt \rfloor$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{r} \right| \leq C n^{-2rH},$$

and consequently

$$\sum_{j,k=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{r} \right| \leq C \lfloor nt \rfloor n^{-2rH}.$$

Proof. For (a), first note that we have $\left|\langle \partial_0, \varepsilon_t \rangle_{\mathfrak{H}}\right| \leq T^H n^{-H}$ by (B.1) and Cauchy-Schwarz. Further, if $\left|\frac{j}{n} - s\right| < \frac{2}{n}$, then by (B.4) we have $\left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_s \right\rangle_{\mathfrak{H}}\right| \leq C n^{-2H}$. Let $\mathcal{J} = \{1 \leq j \leq \lfloor nt \rfloor, |j - ns| > 1\}$; and note that $|\mathcal{J}^c| \leq 2$. Then for the case $r = 1$ we have

$$\begin{aligned}
\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| &= \left|\langle \partial_0, \varepsilon_t \rangle_{\mathfrak{H}}\right| + \sum_{j \in \mathcal{J}^c} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| + \sum_{j \in \mathcal{J}} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| \\
&\leq T^H n^{-H} + C n^{-2H} + C n^{-2H} \sum_{j=1}^{\lfloor nt \rfloor -1} j^{2H-1} + |j - ns|^{2H-1} \\
&\leq C \lfloor nt \rfloor^{2H} n^{-2H} \leq C.
\end{aligned}$$

For the case $r > 1$, we have by (B.4)

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}}^{r} \right| \leq \sup_{0 \leq j \leq \lfloor nt \rfloor} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}}^{r-1} \right| \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| \leq C n^{-2(r-1)H}.$$

For (b), we have by (B.4) and (1)

$$\begin{aligned}
\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}^{r} \right| &\leq \sup_{0 \leq j \leq \lfloor nt \rfloor} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}^{r-1} \right| \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \\
&\leq C n^{-2(r-1)H} \sum_{j=0}^{\lfloor nt \rfloor -1} \frac{1}{2} \left| \mathbb{E}\left[\Delta B_{\frac{j}{n}} \left(B_{\frac{j}{n}} + B_{\frac{j+1}{n}} \right) \right] \right| \\
&= C n^{-2(r-1)H} \sum_{j=0}^{\lfloor nt \rfloor -1} \frac{1}{2} \left| \mathbb{E}\left[B_{\frac{j+1}{n}}^2 - B_{\frac{j}{n}}^2 \right] \right| \\
&= C n^{-2(r-1)H} \sum_{j=0}^{\lfloor nt \rfloor -1} \frac{1}{2} \left[\left(\frac{j+1}{n} \right)^{2H} - \left(\frac{j}{n} \right)^{2H} \right] \\
&\leq C n^{-2(r-1)H} \frac{\lfloor nt \rfloor}{n} \leq C n^{-2(r-1)H}.
\end{aligned}$$

For (c), we note that $\left|\langle \partial_{j/n}, \partial_0 \rangle_{\mathfrak{H}}\right| = \left|\langle \partial_{j/n}, \varepsilon_{1/n} \rangle_{\mathfrak{H}}\right| \le n^{-2H}$. Also note that by (B.1) and Cauchy-Schwarz we have $\left|\langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}}\right| \le n^{-2H}$ for any $1 \le j, k \le \lfloor nt \rfloor$. To begin the proof, we consider the case when $1 \le k \le \lfloor nt \rfloor - 1$ is fixed. Then

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^r \right| \le \sup_{0 \le \jmath \le \lfloor nt \rfloor} \left\{ \sup_{0 \le k \le \lfloor nt \rfloor} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{r-1} \right| \right\} \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right|$$
$$\le n^{-2(r-1)H} \left(n^{-2H} + \sum_{j=1}^{k-2} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| + \sum_{j=k-1}^{k+1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| + \sum_{j=k+2}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| \right)$$

Then we use (B.2) and (B.3) to write

$$n^{-2(r-1)H} \left(n^{-2H} + \sum_{j=1}^{k-2} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| + \sum_{j=k-1}^{k+1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| + \sum_{j=k+2}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}} \right| \right)$$
$$\le n^{2(r-1)H} \left(n^{-2H} + Cn^{-2H} \sum_{j=1}^{k-2} (k-j)^{2H-2} + \sum_{j=k-1}^{k+1} n^{-2H} + Cn^{-2H} \sum_{j=k+2}^{\lfloor nt \rfloor -1} (j-k)^{2H-2} \right)$$
$$\le Cn^{-2rH} \left(4 + 2 \sum_{m=1}^{\infty} m^{2H-2} \right) \le Cn^{-2rH},$$

where we note the sum is finite because $H < 1/2$. For the double sum result we have

$$\sum_{j,k=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^r \right| \le \sum_{k=0}^{\lfloor nt \rfloor -1} \sup_{0 \le k \le \lfloor nt \rfloor} \left\{ \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^r \right| \right\} \le C \lfloor nt \rfloor n^{-2rH}.$$

$\square$

3 Results

3.1 Some results for fBm with $H > 1/14$.

The following proposition summarizes some known results about stochastic integrals with respect to fBm, when the integrals arise from a Riemann sum construction. A comprehensive treatment can be found in an important paper by Gradinaru, Nourdin, Russo & Vallois [3].

Proposition 3.1. *Let $g \in \mathcal{C}^\infty(\mathbb{R})$, such that g and its derivatives have moderate growth. The following Riemann sums converge in probability as $n \to \infty$ to $g(B_t) - g(0)$ for the given ranges of H:*

(a) Midpoint rule: for $1/6 < H < 1/2$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} g'(\widetilde{B}_{\frac{j}{n}}) \Delta B_{\frac{j}{n}},$$

where $\widetilde{B}_{\frac{j}{n}} = \frac{1}{2}\left(B_{\frac{j}{n}} + B_{\frac{j+1}{n}} \right)$.

(b) Trapezoidal rule: For $1/6 < H < 1/2$,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \frac{1}{2} \left(g'(B_{\frac{j}{n}}) + g'(B_{\frac{j+1}{n}}) \right) \Delta B_{\frac{j}{n}}.$$

(c) Simpson's rule: For $1/10 < H < 1/2$,

$$\sum_{j=0}^{\lfloor nt\rfloor-1} \frac{1}{6}\left(g'(B_{\frac{j}{n}}) + 4g'(\widetilde{B}_{\frac{j}{n}}) + g'(B_{\frac{j+1}{n}})\right)\Delta B_{\frac{j}{n}}.$$

(d) Milne's rule: For $1/14 < H < 1/2$,

$$\sum_{j=0}^{\lfloor nt\rfloor-1} \frac{1}{90}\left(7g'(B_{\frac{j}{n}}) + 32g'(B_{\frac{j}{n}} + \frac{1}{4}\Delta B_{\frac{j}{n}}) + 12g'(\widetilde{B}_{\frac{j}{n}}) + 32g'(B_{\frac{j}{n}} + \frac{3}{4}\Delta B_{\frac{j}{n}}) + 7g'(B_{\frac{j+1}{n}})\right)\Delta B_{\frac{j}{n}}.$$

Note that the 'midpoint' sum of part (a) is a different construction than that leading to (6). All of these results follow from Theorem 4.4 of [3], in fact they are also proved there for $H \ge 1/2$. However, here we give a different proof of part (c). By similar techniques, results (a), (b) and (d) could also be done in this way. This proof will contain some results that will be used in Section 3.2, and help set up the proof of Theorem 3.3. We begin with a technical result. The proof of Lemma 3.2 is deferred to Section 4 due to length.

Lemma 3.2. *Let $r = 1, 3, 5, \dots$ and $n \ge 2$ be an integer. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^{2r}$ function such that ϕ and all derivatives up to order $2r$ have moderate growth, and let $\{B_t, t \ge 0\}$ be fBm with Hurst parameter H. Then for each r, there is a constant $C > 0$ such that*

$$\mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt\rfloor-1} \phi(\widetilde{B}_{\frac{j}{n}})\Delta B^r_{\frac{j}{n}}\right)^2\right] \le C \sup_{0\le j\le \lfloor nt\rfloor} \left\|\phi(\widetilde{B}_{\frac{j}{n}})\right\|^2_{\mathbb{D}^{2r,2}} \lfloor nt\rfloor n^{-2rH},$$

where C depends on r and H.

Now for the convergence of the Simpson's rule sum. We begin with some elementary results from the calculus of deterministic functions. For $x, h \in \mathbb{R}$ and a $\mathcal{C}^\infty$ function g, we have the following integral form for the Simpson's rule sum:

$$\begin{aligned} g(x+h) - g(x-h) &= \int_{-h}^{h} g'(x+u)\, du \\ &= \frac{h}{3}\left(g'(x-h) + 4g'(x) + g'(x+h)\right) + \frac{1}{6}\int_0^h \left(g^{(4)}(x-u) - g^{(4)}(x+u)\right)\, u(h-u)^2 du. \end{aligned}$$

See Talman [12] for a nice discussion of the Simpson's rule error term. Next, we consider a Taylor expansion of order 7 for $g^{(4)}$:

$$\begin{aligned} g^{(4)}(x+u) - g^{(4)}(x) &= \sum_{\ell=1}^{6} \frac{g^{(4+\ell)}(x)}{\ell!}u^\ell \;+\; \frac{g^{(11)}(\xi)}{7!}u^7;\ \text{and} \\ g^{(4)}(x) - g^{(4)}(x-u) &= \sum_{\ell=1}^{6} \frac{(-1)^{\ell+1}g^{(4+\ell)}(x)}{\ell!}u^\ell \;+\; \frac{g^{(11)}(\eta)}{7!}u^7 \end{aligned}$$

Adding the above equations, we obtain

$$g^{(4)}(x+u) - g^{(4)}(x-u) = 2\sum_{\nu=1}^{3} \frac{g^{(4+2\nu-1)}(x)}{(2\nu-1)!}u^{2\nu-1} \;+\; \frac{g^{(11)}(\xi) + g^{(11)}(\eta)}{7!}u^7.$$

It follows that we can write

$$g(x+h)-g(x-h)=\frac{h}{3}\left(g'(x-h)+4g'(x)+g'(x+h)\right)-\frac{1}{3}\sum_{\nu=1}^{3}\frac{g^{(4+2\nu-1)}(x)}{(2\nu-1)!}\int_0^h u^{2\nu}(h-u)^2du$$
$$-\frac{g^{(11)}(\xi)+g^{(11)}(\eta)}{(6)(7!)}\int_0^h u^8(h-u)^2du$$
$$=\frac{h}{3}\left(g'(x-h)+4g'(x)+g'(x+h)\right)-\frac{g^{5)}(x)}{90}h^5-A_7g^{(7)}(x)h^7-A_9g^{(9)}(x)h^9$$
$$-\frac{1}{6(7!)}\int_0^h\left[g^{(11)}(\xi)+g^{(11)}(\eta)\right]u^8(h-u)^2du, \tag{*}$$

where A_7, A_9 are positive constants, and $\xi=\xi(u)\in[x-h,x+h]$, with similar for η. With this relation, we now return to Proposition 3.1.c. We begin with the telescoping series,

$$g(B_t)-g(0)=\sum_{j=0}^{\lfloor nt\rfloor-1}\left(g(B_{\frac{j+1}{n}})-g(B_{\frac{j}{n}})\right)+\left(g(B_t)-g(B_{\frac{\lfloor nt\rfloor}{n}})\right)$$
$$=\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{B_{j/n}}^{B_{(j+1)/n}}g'(u)\ du+\left(g(B_t)-g(B_{\frac{\lfloor nt\rfloor}{n}})\right).$$

By continuity, the term $\left(g(B_t)-g(B_{\lfloor nt\rfloor/n})\right)$ tends to zero uniformly on compacts in probability (ucp) as $n\to\infty$, and may be neglected. For each integral term, we use (*) with $x=\widetilde{B}_{j/n}$ and $h=\frac{1}{2}\Delta B_{j/n}$ to obtain

$$\sum_{j=0}^{\lfloor nt\rfloor-1}\int_{B_{j/n}}^{B_{(j+1)/n}}g'(u)\ du=\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{1}{6}\left(g'(B_{\frac{j}{n}})+4g'(\widetilde{B}_{\frac{j}{n}})+g'(B_{\frac{j+1}{n}})\right)-\frac{1}{2^5\ 90}\sum_{j=0}^{\lfloor nt\rfloor-1}g^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^5$$
$$-A_7\sum_{j=0}^{\lfloor nt\rfloor-1}g^{(7)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^7-A_9\sum_{j=0}^{\lfloor nt\rfloor-1}g^{(9)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^9$$
$$-\frac{1}{6(7!)}\sum_{j=0}^{\lfloor nt\rfloor-1}\int_0^{\Delta B_{j/n}}\left(g^{(11)}(\xi)+g^{(11)}(\eta)\right)u^8(\Delta B_{\frac{j}{n}}-u)^2du. \tag{14}$$

By Lemma 3.2, the terms

$$\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{g^{(5)}(\widetilde{B}_{\frac{j}{n}})}{2880}\Delta B_{\frac{j}{n}}^5,\quad A_7\sum_{j=0}^{\lfloor nt\rfloor-1}g^{(7)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^7,\quad A_9\sum_{j=0}^{\lfloor nt\rfloor-1}g^{(9)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^9$$

all tend to zero in $L^2(\Omega)$ as $n\to\infty$. For the last term, we have the $L^2(\Omega)$ estimate

$$\mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt\rfloor-1}\int_0^{\Delta B_{j/n}}\left[g^{(11)}(\xi)+g^{(11)}(\eta)\right]u^8(\Delta B_{\frac{j}{n}}-u)^2du\right)^2\right]$$
$$\leq C\left(\mathbb{E}\left[\sup_{s\in[0,t]}|g^{(11)}(B_s)^4|\right]\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\lfloor nt\rfloor-1}\|\Delta B_{\frac{j}{n}}^{11}\|_{L^4(\Omega)}\right)^2\leq C\lfloor nt\rfloor^2n^{-22H}\leq Cn^{-2H},$$

because $\|\Delta B_{j/n}^{11}\|_{L^4(\Omega)} \leq C\left(\mathbb{E}|\Delta B_{j/n}^2|\right)^{\frac{11}{2}} \leq Cn^{-11H}$ by (B.1) and the Gaussian moments formula. Thus, we have

$$\mathbb{P}\lim_{n\to\infty}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{1}{6}\left(g'(B_{\frac{j}{n}})+4g'(\widetilde{B}_{\frac{j}{n}})+g'(B_{\frac{j+1}{n}})\right)\Delta B_{\frac{j}{n}}=f(B_t)-f(0),$$

when $H > 1/10$, and Proposition 3.1.c is proved. □

As a converse to Proposition 3.1.c (and parts (a), (b) and (d) by similar computation), let $g(x) = f(x)$ be a polynomial such that $g^{(5)} = f^{(5)} = 1$. Then

$$S_n^S(t)=f(B_{\frac{\lfloor nt\rfloor}{n}})-f(0)+\frac{1}{2880}\sum_{j=0}^{\lfloor nt\rfloor-1}\Delta B_{\frac{j}{n}}^5.$$

By Theorem 10 of Nualart and Ortiz-Latorre [10], the sequence $\left(B_t, \sum_{j=0}^{\lfloor nt\rfloor-1}\Delta B_{j/n}^5\right)$ converges in distribution to (B_t, W), where W is a Gaussian random variable, independent of B. It follows that $S_n^S(t)$ does not, in general, converge in probability when $H \leq 1/10$. For the critical case $H = 1/10$, we have the following theorem, which generalizes the result of Theorem 10 of [10] for this particular value of H.

3.2 Main result: fBm with $H = 1/10$.

Throughout the rest of this paper, we will assume that $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function, such that f and all derivatives satisfy moderate growth conditions. Note that this implies $\mathbb{E}\left[\sup_{t\in[0,T]}\left|f^{(n)}(B_t)\right|^p\right] < \infty$ for all $n = 0, 1, 2, \ldots$ and $1 \leq p < \infty$.

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that f and its derivatives have moderate growth conditions, and let $\{B_t, t \geq 0\}$ be a fractional Brownian motion with $H = 1/10$. For $t \geq 0$ and integers $n \geq 2$, Define*

$$S_n^S(t)=\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{1}{6}\left(f'(B_{\frac{j}{n}})+4f'\left((B_{\frac{j}{n}}+B_{\frac{j+1}{n}})/2\right)+f'(B_{\frac{j+1}{n}})\right)\left(B_{\frac{j+1}{n}}-B_{\frac{j}{n}}\right).$$

Then as $n \to \infty$

$$\left(B_t, S_n^S(t)\right)\xrightarrow{\mathcal{L}}\left(B_t, f(B_t)-f(0)+\frac{\beta}{2^5\cdot 90}\int_0^t f^{(5)}(B_s)\,dW_s\right),$$

where $W = \{W_t, t \geq 0\}$ is a Brownian motion, independent of B, and

$$\beta=\sqrt{(5!)2^{-5}\kappa_5+75\kappa_3},\quad \textit{for}\quad \kappa_5=\sum_{p\in\mathbb{Z}}\left((p+1)^{\frac{1}{5}}-2p^{\frac{1}{5}}+(p-1)^{\frac{1}{5}}\right)^5,\ \textit{and}$$

$$\kappa_3=\sum_{p\in\mathbb{Z}}\left((p+1)^{\frac{1}{5}}-2p^{\frac{1}{5}}+(p-1)^{\frac{1}{5}}\right)^3.$$

Consequently,

$$f(B_t)\stackrel{\mathcal{L}}{=}f(0)+\int_0^t f'(B_s)\,d^SB_s-\frac{\beta}{2880}\int_0^t f^{(5)}(B_s)\,dW_s,$$

where $\int_0^t f'(B_s)\,d^SB_s$ denotes the weak limit of the 'Simpson's rule' sum $S_n^S(t)$.

The rest of this section is given to proof of Theorem 3.3, and follows in Sections 3.3 - 3.5. Following the telescoping series argument given in the proof of Proposition 3.1.c (see (14)), we can write

$$f(B_t)-f(0) = S_n^S(t) - \frac{1}{2^5\ 90}\sum_{j=0}^{\lfloor nt\rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^5 - A_7\sum_{j=0}^{\lfloor nt\rfloor -1} f^{(7)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^7 - A_9\sum_{j=0}^{\lfloor nt\rfloor -1} f^{(9)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^9$$
$$-\frac{1}{6(7!)}\sum_{j=0}^{\lfloor nt\rfloor -1}\int_0^{\Delta B_{j/n}}\left(f^{(11)}(\xi)+f^{(11)}(\eta)\right)u^8(\Delta B_{\frac{j}{n}}-u)^2du + \left(f(B_t)-f(B_{\frac{\lfloor nt\rfloor}{n}})\right).$$

As in the proof of Proposition 3.1.c, for $H = 1/10$ it follows from Lemma 3.2 that the terms including $A_7,\ A_9$ and the integral term all tend to zero in $L^2(\Omega)$ as $n\to\infty$, and the term $\left(f(B_t)-f(B_{\lfloor nt\rfloor/n})\right)$ also tends to zero ucp as $n\to\infty$. The main task to prove Theorem 3.3, then, is to show convergence in law of the error term

$$\sum_{j=0}^{\lfloor nt\rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^5. \tag{15}$$

3.3 Malliavin calculus representation.

In order to apply our convergence theorem (Theorem 2.3), we wish to find a Malliavin calculus representation for the term (15). Consider the Hermite polynomial identity $H_5(x) = x^5 - 10H_3(x) - 15x$. Taking $x = \Delta B_{j/n}/\|\Delta B_{j/n}\|_{L^2(\Omega)} = n^H\Delta B_{j/n}$, we have

$$n^{5H}\Delta B_{\frac{j}{n}}^5 = H_5(n^H\Delta B_{\frac{j}{n}}) + 10H_3(n^H\Delta B_{\frac{j}{n}}) + 15n^H\Delta B_{\frac{j}{n}}.$$

Using (8), this gives

$$\sum_{j=0}^{\lfloor nt\rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}^5 = \sum_{j=0}^{\lfloor nt\rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^5(\partial_{\frac{j}{n}}^{\otimes 5})$$
$$+10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^3(\partial_{\frac{j}{n}}^{\otimes 3}) + 15n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}.$$

We first show that the last term tends to zero in $L^1(\Omega)$.

Lemma 3.4. *Under the assumptions of Theorem 3.3, there is a constant $C > 0$ such that*

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}\right)^2\right] \le Cn^{-2H}.$$

Proof. We start with a 2-sided Taylor expansion of $f^{(4)}$ of order 7. That is,

$$f^{(4)}(B_{\frac{j+1}{n}}) - f^{(4)}(\widetilde{B}_{\frac{j}{n}}) = \sum_{\ell=1}^{6}\frac{f^{(4+\ell)}(\widetilde{B}_{\frac{j}{n}})}{2^\ell\ell!}\Delta B_{\frac{j}{n}}^\ell + \frac{f^{(11)}(\xi_j)}{2^7 7!}\Delta B_{\frac{j}{n}}^7$$

and

$$f^{(4)}(\widetilde{B}_{\frac{j}{n}}) - f^{(4)}(B_{\frac{j}{n}}) = \sum_{\ell=1}^{6}\frac{(-1)^{\ell+1}f^{(4+\ell)}(\widetilde{B}_{\frac{j}{n}})}{2^\ell\ell!}\Delta B_{\frac{j}{n}}^\ell + \frac{f^{(11)}(\eta_j)}{2^7 7!}\Delta B_{\frac{j}{n}}^7,$$

for some intermediate values ξ_j, η_j between $B_{j/n}$ and $B_{(j+1)/n}$. Adding the above equations, we obtain

$$f^{(4)}(B_{\frac{j+1}{n}}) - f^{(4)}(B_{\frac{j}{n}}) = f^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}} + \frac{f^{(7)}(\widetilde{B}_{\frac{j}{n}})}{24}\Delta B_{\frac{j}{n}}^3 + \frac{f^{(9)}(\widetilde{B}_{\frac{j}{n}})}{2^4 5!}\Delta B_{\frac{j}{n}}^5 + \frac{f^{(11)}(\xi_j) + f^{(11)}(\eta_j)}{2^7 7!}\Delta B_{\frac{j}{n}}^7. \quad (16)$$

It follows that we can write

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\Delta B_{\frac{j}{n}}\right)^2\right] \le 4\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\left(f^{(4)}(B_{\frac{j+1}{n}}) - f^{(4)}(B_{\frac{j}{n}})\right)\right)^2\right] + 4\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(7)}(\widetilde{B}_{\frac{j}{n}})}{24}\Delta B_{\frac{j}{n}}^3\right)^2\right] + 4\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(9)}(\widetilde{B}_{\frac{j}{n}})}{2^4 5!}\Delta B_{\frac{j}{n}}^5\right)^2\right] + 4\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(11)}(\xi_j) + f^{(11)}(\eta_j)}{2^7 7!}\Delta B_{\frac{j}{n}}^7\right)^2\right].$$

By growth assumptions on $f^{(4)}$,

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\left(f^{(4)}(B_{\frac{j+1}{n}}) - f^{(4)}(B_{\frac{j}{n}})\right)\right)^2\right] = n^{-8H}\mathbb{E}\left[\left(f^{(4)}(B_{\frac{\lfloor nt\rfloor}{n}}) - f^{(4)}(0)\right)^2\right] \le Cn^{-8H}.$$

By Lemma 3.2,

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(7)}(\widetilde{B}_{\frac{j}{n}})}{24}\Delta B_{\frac{j}{n}}^3\right)^2\right] \le C\sup_{0\le j\le\lfloor nt\rfloor}\|f^{(7)}(\widetilde{B}_{\frac{j}{n}})\|_{\mathbb{D}^{6,2}}^2\lfloor nt\rfloor n^{-14H},$$

and

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(9)}(\widetilde{B}_{\frac{j}{n}})}{2^4 5!}\Delta B_{\frac{j}{n}}^5\right)^2\right] \le C\sup_{0\le j\le\lfloor nt\rfloor}\|f^{(9)}(\widetilde{B}_{\frac{j}{n}})\|_{\mathbb{D}^{10,2}}^2\lfloor nt\rfloor n^{-18H}.$$

Then by (B.1),

$$\mathbb{E}\left[\left(n^{-4H}\sum_{j=0}^{\lfloor nt\rfloor-1}\frac{f^{(11)}(\xi_j) + f^{(11)}(\eta_j)}{2^7 7!}\Delta B_{\frac{j}{n}}^7\right)^2\right] \le C\left(\mathbb{E}\left[\sup_{s\in[0,t]}\left|f^{(11)}(B_s)^4\right|\right]\right)^{\frac{1}{2}} n^{-8H}\left(\sum_{j=0}^{\lfloor nt\rfloor-1}\|\Delta B_{\frac{j}{n}}^7\|_{L^4(\Omega)}\right)^2 \le C\lfloor nt\rfloor^2 n^{-22H} \le Cn^{-2H}.$$

This proves the lemma. □

Lemma 3.4 shows that only the terms

$$\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^5\left(\partial_{\frac{j}{n}}^{\otimes 5}\right) + 10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^3(\partial_{\frac{j}{n}}^{\otimes 3})$$

are significant. Using Lemma 2.1.a, we can write the first term as

$$\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^5\left(f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}\right)+\sum_{r=1}^{5}\binom{5}{r}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^{5-r}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(5-r)}\right)\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r}.$$

By Lemma 2.1.c and (B.1), we have the estimate

$$\left\|\delta^{(5-r)}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(5-r)}\right)\right\|_{L^2(\Omega)}\le C\left\|\partial_{\frac{j}{n}}^{\otimes(5-r)}\right\|_{\mathfrak{H}^{\otimes 5-r}}\le Cn^{(r-5)H}.$$

It follows that for $r=1,\dots,5$, we can use Lemma 2.6.b,

$$\mathbb{E}\left|\binom{5}{r}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^{(5-r)}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(5-r)}\right)\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|$$
$$\le Cn^{(r-5)H}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|\le Cn^{-(3+r)H}.$$

By a similar computation,

$$\begin{aligned}10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^3(\partial_{\frac{j}{n}}^{\otimes 3})&=10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^3\left(f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}\right)\\&\quad+10n^{-2H}\sum_{r=1}^{3}\binom{3}{r}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^{(3-r)}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(3-r)}\right)\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r},\end{aligned}$$

where

$$n^{-2H}\mathbb{E}\left|\sum_{r=1}^{3}\binom{3}{r}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^{(3-r)}\left(f^{(5+r)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes(3-r)}\right)\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{r}\right|\le Cn^{-4H}.$$

Therefore, we define

$$F_n:=\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^5\left(f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}\right)=\delta^5(u_n),\quad\text{where}\ \ u_n=\sum_{j=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5};\ \text{and}$$
$$G_n:=10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1}\delta^3\left(f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}\right)=\delta^3(v_n),\quad\text{where}\ \ v_n=10n^{-2H}\sum_{j=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}.$$

It follows that for large n, the term (15) may be represented as $F_n+G_n+\epsilon_n$, where $\epsilon_n\to 0$ in $L^1(\Omega)$. Then, as introduced in Remark 2.5, we will work with the vector sequence (F_n,G_n).

3.4 Conditions of Theorem 2.3.

Our main task in this step is to show that the sequence of random vectors (F_n,G_n) satisfies the conditions of Theorem 2.3. The first condition is that (F_n,G_n) is bounded in $L^1(\Omega)$. In fact, we have a stronger result that will also be helpful with later conditions.

Lemma 3.5. *Fix real numbers $0 < t \le T$ and $p \ge 2$, and integer $n \ge 2$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that ϕ and all its derivatives have moderate growth. For integer $1 \le q \le 5$, define*

$$w_n = \sum_{j=0}^{\lfloor nt \rfloor - 1} \phi(\widetilde{B}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes q}.$$

Then for integers $0 \le a \le 5$, there exists a constant $c_{q,a}$ such that

$$\|D^a \delta^q(w_n)\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \le c_{q,a} \sup_{0\le j\le \lfloor nt \rfloor} \left\| \phi(\widetilde{B}_{\frac{j}{n}}) \right\|^2_{\mathbb{D}^{q+a,p}} \lfloor nt \rfloor n^{-2qH} \le C n^{1-2qH}.$$

In particular,

$$\|D^a F_n\|_{L^p(\Omega;\mathfrak{H}^{\otimes a})} + \|D^a G_n\|_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \le C. \tag{17}$$

Proof. This proof follows a similar result in [6], see Theorem 5.2. First, note that by Lemma 2.6.c and growth conditions on ϕ, for each integer $b \ge 0$,

$$\begin{aligned}
\left\| D^b w_n \right\|^2_{\mathfrak{H}^{\otimes q+b}} &= \left\| \sum_{j=0}^{\lfloor nt \rfloor -1} \phi^{(b)}(\widetilde{B}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes q} \otimes \widetilde{\varepsilon}_{\frac{j}{n}}^{\otimes b} \right\|^2_{\mathfrak{H}^{\otimes q+b}} \\
&\le \sup_{0\le j\le \lfloor nt \rfloor} \left| \phi^{(b)}(\widetilde{B}_{\frac{j}{n}}) \right|^2 \sup_{0\le j,k\le \lfloor nt \rfloor} \left| \left\langle \widetilde{\varepsilon}_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}} \right\rangle^b \right| \sum_{j,k=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle^q_{\mathfrak{H}} \right| \\
&\le C \lfloor nt \rfloor n^{-2qH} \sup_{0\le j\le \lfloor nt \rfloor} \left| \phi^{(b)}(\widetilde{B}_{\frac{j}{n}}) \right|^2.
\end{aligned}$$

It follows that for $p \ge 2$,

$$\left\| D^b w_n \right\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes q+b})} \le C \lfloor nt \rfloor n^{-2qH} \mathbb{E}\left[\sup_{0\le j\le \lfloor nt \rfloor} \left| \phi^{(b)}(\widetilde{B}_{\frac{j}{n}}) \right|^p \right]^{\frac{2}{p}}.$$

Then, using the Meyer inequality (see [6], Proposition 1.5.7),

$$\|D^a \delta^q(w_n)\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \le \|\delta^q(w_n)\|^2_{\mathbb{D}^{a,p}} \le C \lfloor nt \rfloor n^{-2qH} \sup_{0\le j\le \lfloor nt \rfloor} \left\| \phi(\widetilde{B}_{\frac{j}{n}}) \right\|^2_{\mathbb{D}^{q+a,p}(\mathfrak{H}^q)} \le C \lfloor nt \rfloor n^{-2qH}. \tag{18}$$

For (17), we have

$$\|D^a F_n\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} = \left\| D^a \delta^5(u_n) \right\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \le C \lfloor nt \rfloor n^{-10H} \sup_{0\le j\le \lfloor nt \rfloor} \left\| f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \right\|^2_{\mathbb{D}^{5+a,p}(\mathfrak{H}^{\otimes 5})} \le C,$$

and

$$\|D^a G_n\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} = \left\| n^{-2H} D^a \delta^3(u_n) \right\|^2_{L^p(\Omega;\mathfrak{H}^{\otimes a})} \le C \lfloor nt \rfloor n^{-10H} \sup_{0\le j\le \lfloor nt \rfloor} \left\| f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \right\|^2_{\mathbb{D}^{3+a,p}(\mathfrak{H}^{\otimes 3})} \le C.$$

□

The fact that (F_n, G_n) is bounded in $L^1(\Omega)$ follows by taking $a = 0$. Next, we consider condition (a) of Theorem 2.3.

Lemma 3.6. *Under the assumptions of Theorem 3.3, (F_n, G_n) satisfies condition (a) of Theorem 2.3. That is, we have*

(a) For arbitrary $h \in \mathfrak{H}^{\otimes 5}$ and $g \in \mathfrak{H}^{\otimes 3}$,

$$\lim_{n\to\infty} \mathbb{E}\left|\langle u_n, h\rangle_{\mathfrak{H}^{\otimes 5}}\right| = \lim_{n\to\infty} \mathbb{E}\left|\langle v_n, g\rangle_{\mathfrak{H}^{\otimes 3}}\right| = 0.$$

(b) $\lim_{n\to\infty} \mathbb{E}\left|\left\langle u_n, \bigotimes_{i=1}^{s} D^{a_i} F_n \bigotimes_{i=s+1}^{r} D^{a_i} G_n \otimes h\right\rangle_{\mathfrak{H}^{\otimes 5}}\right| = 0$*, where* $0 \le a_i < 5$*,* $1 \le a_1 + \cdots + a_r < 5$*, and* $h \in \mathfrak{H}^{\otimes 5-(a_1+\cdots+a_r)}$*; and* $\lim_{n\to\infty} \mathbb{E}\left|\left\langle v_n, \bigotimes_{i=1}^{s} D^{b_i} F_n \bigotimes_{i=s+1}^{r} D^{b_i} G_n \otimes g\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| = 0$*, where* $0 \le b_i < 3$*,* $1 \le b_1 + \cdots + b_r < 3$*, and* $g \in \mathfrak{H}^{\otimes 3-(b_1+\cdots+b_r)}$.

(c) $\lim_{n\to\infty} \mathbb{E}\left|\left\langle u_n, \bigotimes_{i=1}^{s} D^{a_i} F_n \bigotimes_{i=s+1}^{r} D^{a_i} G_n\right\rangle_{\mathfrak{H}^{\otimes 5}}\right| = 0$*, where* $r \ge 2$*,* $0 \le a_i < 5$ *and* $a_1+\cdots+a_r = 5$*; and*
$\lim_{n\to\infty} \mathbb{E}\left|\left\langle v_n, \bigotimes_{i=1}^{s} D^{b_i} F_n \bigotimes_{i=s+1}^{r} D^{b_i} G_n\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| = 0$*, where* $r \ge 2$*,* $0 \le b_i < 3$ *and* $b_1+\cdots+b_r = 3$*.*

The proof of this lemma is deferred to Section 4 due to its length. To verify condition (b) of Theorem 2.3, we have four terms to consider:

- $\left\langle u_n, D^5 G_n\right\rangle_{\mathfrak{H}^{\otimes 5}}$
- $\left\langle v_n, D^3 F_n\right\rangle_{\mathfrak{H}^{\otimes 3}}$
- $\left\langle u_n, D^5 F_n\right\rangle_{\mathfrak{H}^{\otimes 5}}$
- $\left\langle v_n, D^3 G_n\right\rangle_{\mathfrak{H}^{\otimes 3}}$

We deal with the first two terms in the following lemma. The proof is given in Section 4.

Lemma 3.7. *Under the assumptions of Theorem 3.3, we have*

(a) $\lim_{n\to\infty} \mathbb{E}\left|\left\langle u_n, D^5 G_n\right\rangle_{\mathfrak{H}^{\otimes 5}}\right| = 0$

(b) $\lim_{n\to\infty} \mathbb{E}\left|\left\langle v_n, D^3 F_n\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| = 0.$

This leaves the variance terms. Lemma 2.1.b allows us to write

$$\begin{aligned}
\left\langle u_n, D^5 F_n\right\rangle_{\mathfrak{H}^{\otimes 5}} &= \sum_{j,k=0}^{\lfloor nt\rfloor-1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, D^5\delta^5\left(f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right)\right\rangle_{\mathfrak{H}^{\otimes 5}} \\
&= \sum_{z=0}^{4}\binom{5}{z}^2 z! \sum_{j,k=0}^{\lfloor nt\rfloor-1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, \delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-z}\right)\partial_{\frac{k}{n}}^{\otimes z}\otimes \widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 5-z}\right\rangle_{\mathfrak{H}^{\otimes 5}} \\
&\quad + 5! \sum_{j,k=0}^{\lfloor nt\rfloor-1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right\rangle_{\mathfrak{H}^{\otimes 5}}.
\end{aligned}$$

We first deal with the case $0 \le z \le 4$. We have

$$\begin{aligned}
&\mathbb{E}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, \delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-z}\right)\partial_{\frac{k}{n}}^{\otimes z}\otimes \widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 5-z}\right\rangle_{\mathfrak{H}^{\otimes 5}}\right| \\
&\qquad \le C \sup_{0\le j\le \lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)} \sup_{0\le k\le \lfloor nt\rfloor}\left\|\delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-z}\right)\right\|_{L^2(\Omega)} \\
&\qquad\qquad \times \sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5-z}\right|.
\end{aligned}$$

By (B.1) and Lemma 2.1.c, we have

$$\sup_{0\le k\le \lfloor nt\rfloor} \left\| \delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}}) \partial_{\frac{k}{n}}^{\otimes 5-z}\right)\right\|_{L^2(\Omega)} \le C\|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^{5-z} \le Cn^{-(5-z)H},$$

so for the case $z=0$, we have

$$\begin{aligned}
\sup_{0\le j\le \lfloor nt\rfloor} \left\| f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)} &\sup_{0\le k\le \lfloor nt\rfloor} \left\| \delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}}) \partial_{\frac{k}{n}}^{\otimes 5-z}\right)\right\|_{L^2(\Omega)} \\
&\times \sum_{j,k=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z} \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5-z}\right| \\
&\le Cn^{-5H} \sup_{0\le j\le \lfloor nt\rfloor}\left\{ \sup_{s\in[0,t]} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s\right\rangle_{\mathfrak{H}}^{4}\right|\right\} \sum_{k=0}^{\lfloor nt\rfloor -1} \sup_{0\le k\le \lfloor nt\rfloor} \sum_{j=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|.
\end{aligned}$$

By (B.4) and Lemma 2.6.a, respectively,

$$\sup_{0\le j\le \lfloor nt\rfloor}\left\{ \sup_{s\in[0,t]} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s\right\rangle_{\mathfrak{H}}^{4}\right|\right\} \le Cn^{-8H} \quad\text{and}\quad \sup_{0\le k\le \lfloor nt\rfloor} \sum_{j=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right| \le C,$$

so this gives

$$Cn^{-5H} \sup_{0\le j\le \lfloor nt\rfloor}\left\{ \sup_{s\in[0,t]} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s\right\rangle_{\mathfrak{H}}^{4}\right|\right\} \sum_{k=0}^{\lfloor nt\rfloor -1} \sup_{0\le k\le \lfloor nt\rfloor} \sum_{j=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right| \le C\lfloor nt\rfloor n^{-13H} \le Cn^{-3H}.$$

If $1\le z\le 4$, then by (B.1), (B.4) and Lemma 2.6.c we have an upper bound of

$$\begin{aligned}
&\sup_{0\le j\le \lfloor nt\rfloor} \left\| f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)} \sup_{0\le k\le \lfloor nt\rfloor} \left\| \delta^{5-z}\left(f^{(10-z)}(\widetilde{B}_{\frac{k}{n}}) \partial_{\frac{k}{n}}^{\otimes 5-z}\right)\right\|_{L^2(\Omega)} \sum_{j,k=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z} \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{5-z}\right| \\
&\quad \le C\|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^{5-z} \sup_{0\le j\le \lfloor nt\rfloor}\left\{ \sup_{s\in[0,t]} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_s\right\rangle^{5-z}\right|\right\} \sum_{j,k=0}^{\lfloor nt\rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\right| \le C\lfloor nt\rfloor n^{-(15-z)H} \le Cn^{-H},
\end{aligned}$$

because $z<5$. It follows that the term corresponding to each $z=0,\dots,4$ vanishes in $L^1(\Omega)$, and we have that only the term with $z=5$ is significant. For the case $z=5$, we use a result from [6], see proof of Theorem 5.2.

$$\begin{aligned}
5! &\sum_{j,k=0}^{\lfloor nt\rfloor -1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 5}, f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right\rangle_{\mathfrak{H}^{\otimes 5}} \\
&= 5! \sum_{j,k=0}^{\lfloor nt\rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) f^{(5)}(\widetilde{B}_{\frac{k}{n}}) \left(\mathbb{E}\left[\Delta B_{\frac{j}{n}}, \Delta B_{\frac{k}{n}}\right]\right)^5 \\
&= \frac{5!}{2^5 n^{10H}} \sum_{p=-\infty}^{\infty} \sum_{j=(0\vee -p)}^{(\lfloor nt\rfloor -1)\wedge(\lfloor nt\rfloor -1-p)} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) f^{(5)}(\widetilde{B}_{\frac{j+p}{n}}) \left(|p+1|^{2H} - 2|p|^{2H} + |p-1|^{2H}\right)^5,
\end{aligned}$$

which (for $H=1/10$) converges in $L^1(\Omega)$ to

$$\frac{5!}{2^5}\kappa_5 \int_0^t f^{(5)}(B_s)^2\, ds, \text{ where } \kappa_5 = \sum_{p\in\mathbb{Z}} \left(|p+1|^{\frac{1}{5}} - 2|p|^{\frac{1}{5}} + |p-1|^{\frac{1}{5}}\right)^5. \tag{19}$$

Hence, we have that

$$\lim_{n\to\infty}\left\langle u_n, D^5F_n\right\rangle_{\mathfrak{H}^{\otimes 5}} = \frac{5!}{2^5}\kappa_5\int_0^t f^{(5)}(B_s)^2 ds. \tag{20}$$

Similarly, we have

$$\left\langle v_n, D^3G_n\right\rangle_{\mathfrak{H}^{\otimes 3}} = 10^2 n^{-4H}\sum_{z=0}^{3}\binom{3}{z}^2 z!\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}, \delta^{3-z}\left(f^{(8-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3-z}\right)\partial_{\frac{k}{n}}^{\otimes z}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 3-z}\right\rangle_{\mathfrak{H}^{\otimes 3}}.$$

For $z=0$,

$$\begin{aligned}
&100n^{-4H}\mathbb{E}\left|\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}, \delta^3\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3}\right)\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| \\
&\quad\le 100n^{-4H}\sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\le k\le\lfloor nt\rfloor}\left\|\delta^3\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3}\right)\right\|_{L^2(\Omega)}\sup_{j,k}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^2\right| \\
&\qquad\times\sum_{k=0}^{\lfloor nt\rfloor-1}\sup_{s\in[0,t]}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\varepsilon_s\right\rangle_{\mathfrak{H}}\right| \\
&\quad\le C\lfloor nt\rfloor n^{-11H}\le Cn^{-H}.
\end{aligned}$$

For $z=1$ or $z=2$, by (B.4) and Lemma 2.6.c,

$$\begin{aligned}
&100\binom{3}{z}^2 z! n^{-4H}\mathbb{E}\left|\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}, \delta^{3-z}\left(f^{(8-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3-z}\right)\partial_{\frac{k}{n}}^{\otimes z}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 3-z}\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| \\
&\quad\le Cn^{-4H}\sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\le k\le\lfloor nt\rfloor}\left\|\delta^{3-z}\left(f^{(8-z)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3-z}\right)\right\|_{L^2(\Omega)}\sup_{j,k}\left|\left\langle\partial_{\frac{j}{n}},\widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3-z}\right| \\
&\qquad\times\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\right| \\
&\quad\le C\lfloor nt\rfloor n^{-(13-z)H}\le Cn^{-H},
\end{aligned}$$

because $z\le 2$. Then for $z=3$, we have

$$\begin{aligned}
&600n^{-4H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}, f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 3}\right\rangle_{\mathfrak{H}^{\otimes 3}} \\
&\qquad=\frac{600}{2^3 n^{10H}}\sum_{j,k=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})f^{(5)}(\widetilde{B}_{\frac{k}{n}})\left(|j-k+1|^{2H}-2|j-k|^{2H}+|j-k-1|^{2H}\right)^3.
\end{aligned}$$

Similar to (19), this converges in $L^1(\Omega)$ to

$$75\kappa_3\int_0^t f^{(5)}(B_s)^2\,ds,\ \text{where}\ \kappa_3=\sum_{p\in\mathbb{Z}}\left(|p+1|^{\frac{1}{5}}-2|p|^{\frac{1}{5}}+|p-1|^{\frac{1}{5}}\right)^3. \tag{21}$$

Hence, we have that

$$\lim_{n\to\infty}\left\langle v_n, D^3G_n\right\rangle_{\mathfrak{H}^{\otimes 3}} = 75\kappa_3\int_0^t f^{(5)}(B_s)^2 ds. \tag{22}$$

3.5 Proof of Theorem 3.3.

By Sections 3.3, the term (15) is dominated in probability by $\frac{1}{2880}(F_n + G_n)$. By the results of Section 3.4, the vector (F_n, G_n) satisfies Theorem 2.3, that is, (F_n, G_n) converges stably as $n \to \infty$ to a mean-zero Gaussian random vector (F_∞, G_∞) with independent components, whose variances are given by (20) and (22), respectively. It follows that $F_n + G_n$ converges in distribution to a centered Gaussian random variable with variance

$$s^2 = \frac{5!}{2^5}\kappa_5 \int_0^t f^{(5)}(B_s)^2 \, ds + 75\kappa_3 \int_0^t f^{(5)}(B_s)^2 \, ds = \beta^2 \int_0^t f^{(5)}(B_s)^2 \, ds,$$

where $\beta^2 = (5!)2^{-5}\kappa_5 + 75\kappa_3$. The result of Theorem 3.3 then follows from the Itô isometry. This concludes the proof.

4 Proof of Technical Lemmas

4.1 Proof of Lemma 3.2

To simplify notation, let $Y_j := \phi(\widetilde{B}_{\frac{j}{n}})$. Note that by (B.1), we have $\|\Delta B_{\frac{j}{n}}\|_{L^2(\Omega)} = \|\partial_{\frac{j}{n}}\|_{\mathfrak{H}} = n^{-H}$. For Hermite polynomials $H_r(x)$, $r \geq 1$, it can be shown by induction on the relation $H_{q+1}(x) = xH_q(x) - qH_{q-1}(x)$ that

$$x^r = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} C(r,p) H_{r-2p}(x),$$

where each $C(r,p)$ is an integer constant. From Section 2.1, we use (8) with $x = \Delta B_{\frac{j}{n}} / \|\Delta B_{\frac{j}{n}}\|_{L^2(\Omega)} = n^H \Delta B_{\frac{j}{n}}$ to write

$$H_r\left(n^H \Delta B_{\frac{j}{n}}\right) = \delta^r\left(n^{rH} \partial_{\frac{j}{n}}^{\otimes r}\right).$$

It follows that

$$n^{rH} \Delta B_{\frac{j}{n}}^r = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} C(r,p) H_{r-2p}(n^H \Delta B_{\frac{j}{n}}) = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} C(r,p) \delta^{r-2p}\left(n^{(r-2p)H} \partial_{\frac{j}{n}}^{\otimes r-2p}\right),$$

which implies

$$\Delta B_{\frac{j}{n}}^r = \sum_{p=0}^{\lfloor \frac{r}{2} \rfloor} C(r,p) n^{-2pH} \delta^{r-2p}\left(\partial_{\frac{j}{n}}^{\otimes r-2p}\right).$$

With this representation for $\Delta B_{j/n}^r$, we then have

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=0}^{\lfloor nt \rfloor - 1} Y_j \Delta B_{\frac{j}{n}}^r\right)^2\right] \\
&\quad = \sum_{p,p'=0}^{\lfloor \frac{r}{2} \rfloor} C(r,p)C(r,p') n^{-2H(p+p')} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \mathbb{E}\left[Y_j Y_k \delta^{r-2p}\left(\partial_{\frac{j}{n}}^{\otimes r-2p}\right) \delta^{r-2p'}\left(\partial_{\frac{k}{n}}^{\otimes r-2p'}\right)\right] \\
&\quad \leq \sum_{p,p'=0}^{\lfloor \frac{r}{2} \rfloor} |C(r,p)C(r,p')| n^{-2H(p+p')} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left|\mathbb{E}\left[Y_j Y_k \delta^{r-2p}\left(\partial_{\frac{j}{n}}^{\otimes r-2p}\right) \delta^{r-2p'}\left(\partial_{\frac{k}{n}}^{\otimes r-2p'}\right)\right]\right|. \qquad (23)
\end{aligned}$$

By Lemma 2.1.d, the product
$$\delta^{r-2p}\left(\partial_{\frac{j}{n}}^{\otimes r-2p}\right)\delta^{r-2p'}\left(\partial_{\frac{k}{n}}^{\otimes r-2p'}\right)$$
consists of terms of the form
$$C\delta^{2r-2(p+p')-2z}\left(\partial_{\frac{j}{n}}^{\otimes r-2p-z}\otimes\partial_{\frac{k}{n}}^{\otimes r-2p'-z}\right)\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}, \tag{24}$$
where $z\geq 0$ is an integer satisfying $2r-2(p+p')-2z\geq 0$. Using (24), we can write that (23) consists of nonnegative terms of the form
$$Cn^{-2H(p+p')}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\mathbb{E}\left[Y_jY_k\delta^{2r-2(p+p')-2z}\left(\partial_{\frac{j}{n}}^{\otimes r-2p-z}\otimes\partial_{\frac{k}{n}}^{\otimes r-2p'-z}\right)\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\right]\right|. \tag{25}$$
To address terms of this type, suppose first that $z\geq 1$. Lemma 2.1.c implies that
$$\begin{aligned}\left\|\delta^{2r-2(p+p')-2z}\left(\partial_{\frac{j}{n}}^{\otimes r-2p-z}\otimes\partial_{\frac{k}{n}}^{\otimes r-2p'-z}\right)\right\|_{L^2(\Omega)}&\leq C\left(\|\partial_{\frac{j}{n}}\|_{\mathfrak{H}}^{r-2p-z}\|\partial_{\frac{k}{n}}\|_{\mathfrak{H}}^{r-2p'-z}\right)\\&\leq C\left\|\partial_{\frac{1}{n}}\right\|_{\mathfrak{H}}^{2r-2(p+p')-2z}=Cn^{-2H(r-p-p'-z)}.\end{aligned}$$
Hence, for $z\geq 1$, (25) is bounded by
$$\begin{aligned}Cn^{-2H(p+p')}\sup_{0\leq j\leq\lfloor nt\rfloor}\|Y_j\|_{L^2(\Omega)}^2\left\|\partial_{\frac{1}{n}}\right\|_{\mathfrak{H}}^{2r-2(p+p')-2z}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{z}\right|\\\leq C\sup_{0\leq j\leq\lfloor nt\rfloor}\|Y_j\|_{\mathbb{D}^{2r,2}}^2\lfloor nt\rfloor n^{-2rH},\end{aligned}$$
which follows from Lemma 2.6.c.

On the other hand, for the terms with $z=0$, by (10) we have
$$\begin{aligned}&\mathbb{E}\left[Y_jY_k\delta^{2r-2(p+p')}\left(\partial_{\frac{j}{n}}^{\otimes r-2p}\otimes\partial_{\frac{k}{n}}^{\otimes r-2p'}\right)\right]\\&\qquad=\mathbb{E}\left\langle D^{2r-2(p+p')}Y_jY_k,\partial_{\frac{j}{n}}^{\otimes r-2p}\otimes\partial_{\frac{k}{n}}^{\otimes r-2p'}\right\rangle_{\mathfrak{H}^{\otimes 2r-2(p+p')}}.\end{aligned} \tag{26}$$
By definition of the Malliavin derivative and Leibniz rule, $D^{2r-2(p+p')}Y_jY_k$ consists of terms of the form $D^aY_j\otimes D^bY_k$, where $a+b=2r-2(p+p')$. Without loss of generality, we may assume $b\geq 1$. By assumptions on ϕ and the definition of the Malliavin derivative, we know that $D^bY_k=\phi^{(b)}(\widetilde{B}_{k/n})\widetilde{\varepsilon}_{k/n}^{\otimes b}$, and we know that for each $b\leq 2r$, $D^bY_k\in L^2(\Omega;\mathfrak{H}^{\otimes b})$. It follows that we can write,
$$\begin{aligned}&\left|\mathbb{E}\left\langle D^aY_j\otimes D^bY_k,\partial_{\frac{j}{n}}^{\otimes r-2p}\otimes\partial_{\frac{k}{n}}^{\otimes r-2p'}\right\rangle_{\mathfrak{H}^{\otimes a+b}}\right|\\&\qquad\leq C\|Y_j\|_{\mathbb{D}^{2r,2}}\|Y_k\|_{\mathbb{D}^{2r,2}}\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{\phi}\right|\left|\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{a-\phi}\right|\\&\qquad\qquad\times\left|\left\langle\widetilde{\varepsilon}_{\frac{k}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}^{\psi}\right|\left|\left\langle\widetilde{\varepsilon}_{\frac{k}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{b-\psi}\right|,\end{aligned}$$
for integers $0\leq\phi\leq a$, $0\leq\psi\leq b$. Without loss of generality, we may assume $\psi\geq 1$, and by implication $b\geq 1$. Then using (B.4),
$$\left|\mathbb{E}\left\langle D^aY_jD^bY_k,\partial_{\frac{j}{n}}^{\otimes r-2p}\otimes\partial_{\frac{k}{n}}^{\otimes r-2p'}\right\rangle_{\mathfrak{H}^{\otimes a+b}}\right|\leq C\sup_{0\leq j\leq\lfloor nt\rfloor}\|Y_j\|_{\mathbb{D}^{2r,2}}^2n^{-2H(a+b-1)}\left|\left\langle\widetilde{\varepsilon}_{\frac{k}{n}},\partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|.$$

Thus, for each pair (a, b), the corresponding term of (25) is bounded by

$$Cn^{-2H(p+p')} \sum_{j,k=0}^{\lfloor nt \rfloor -1} \left| \mathbb{E}\left[Y_j Y_k \delta^{2r-2(p+p')} \left(\partial_{\frac{j}{n}}^{\otimes r-2p} \otimes \partial_{\frac{k}{n}}^{\otimes r-2p'} \right) \right] \right|$$

$$\leq Cn^{-2H(p+p'+a+b-1)} \sup_{0 \leq j \leq \lfloor nt \rfloor} \|Y_j\|_{\mathbb{D}^{2r,2}}^2 \sum_{j,k=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right|$$

$$\leq Cn^{-2H(p+p'+a+b-1)} \sup_{0 \leq j \leq \lfloor nt \rfloor} \|Y_j\|_{\mathbb{D}^{2r,2}}^2 \sum_{j,k=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right|.$$

By Lemma 2.6.a,

$$\sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \leq C \lfloor nt \rfloor^{2H} n^{-2H} \leq C$$

for all $0 \leq k \leq \lfloor nt \rfloor$, so that

$$Cn^{-2H(p+p'+a+b-1)} \sup_{0 \leq j \leq \lfloor nt \rfloor} \|Y_j\|_{\mathbb{D}^{2r,2}}^2 \sum_{k=0}^{\lfloor nt \rfloor -1} \left\{ \sup_{0 \leq k \leq \lfloor nt \rfloor} \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \widetilde{\varepsilon}_{\frac{k}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \right\}$$

$$\leq C \sup_{0 \leq j \leq \lfloor nt \rfloor} \|Y_j\|_{\mathbb{D}^{2r,2}}^2 \lfloor nt \rfloor n^{-2H(p+p'+a+b-1)},$$

where $p + p' + a + b - 1 = 2r - (p + p') - 1 \geq r$, since $p + p' + 1 \leq 2 \left\lfloor \frac{r}{2} \right\rfloor + 1 \leq r$, for odd integer r. This concludes the proof.

4.2 Proof of Lemma 3.6.

For $\theta \in \{0, 2\}$ define

$$w_n(\theta) = n^{-\theta H} \sum_{j=0}^{\lfloor nt \rfloor -1} f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 5-\theta}; \quad \text{and} \quad \Phi_n(\theta) = \delta^{5-\theta}(w_n(\theta)).$$

This allows us to write $u_n = w_n(0)$, $F_n = \Phi_n(0)$, $v_n = 10w_n(2)$, and $G_n = 10\Phi_n(2)$. Following Remark 2.4, we may assume that $h \in \mathfrak{H}^{\otimes 5-\theta}$ has the form $\varepsilon_{t_1} \otimes \cdots \otimes \varepsilon_{t_{5-\theta}}$, for some set of times $\{t_1, \ldots, t_{5-\theta}\}$ in $[0, T]^{5-\theta}$. Then for (a), using (B.4) and Lemma 2.6.a,

$$\mathbb{E} \left| \langle w_n(\theta), h \rangle_{\mathfrak{H}^{\otimes 5-\theta}} \right| = n^{-\theta H} \mathbb{E} \left| \sum_{j=0}^{\lfloor nt \rfloor -1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 5-\theta}, \varepsilon_{t_1} \otimes \cdots \otimes \varepsilon_{t_{5-\theta}} \right\rangle_{\mathfrak{H}^{\otimes 5-\theta}} \right|$$

$$\leq n^{-\theta H} \mathbb{E} \left[\sup_{s \in [0,t]} \left| f^{(5)}(B_s) \right| \right] \sum_{j=0}^{\lfloor nt \rfloor -1} \prod_{k=1}^{5-\theta} \left| \left\langle \partial_{\frac{j}{n}}, \varepsilon_{t_k} \right\rangle_{\mathfrak{H}} \right|$$

$$\leq Cn^{-(8-\theta)H} \leq Cn^{-6H},$$

where the last inequality follows because $\theta \leq 2$.

Next, for (b), consider integers $0 \leq a_i < 5 - \theta$, $0 \leq s \leq r < 5 - \theta$, $r \geq 1$ and q, such that $s \leq r$,

$1 \le a_1 + \cdots + a_r < 5 - \theta$ and $q = 5 - \theta - (a_1 + \cdots + a_r) \ge 1$. We have

$$\mathbb{E}\left|\left\langle w_n(\theta), \bigotimes_{i=1}^{s} D^{a_i} F_n \bigotimes_{i=s+1}^{r} D^{a_i} G_n \otimes h \right\rangle_{\mathfrak{H}^{\otimes 5-\theta}}\right|$$
$$\le n^{-\theta H}\mathbb{E} \sum_{j=0}^{\lfloor nt \rfloor -1} \left| f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \prod_{i=1}^{s} \left\langle \partial_{\frac{j}{n}}^{\otimes a_i}, D^{a_i} F_n \right\rangle_{\mathfrak{H}^{\otimes a_i}} \left(\prod_{i=s+1}^{r} \left\langle \partial_{\frac{j}{n}}^{\otimes a_i}, D^{a_i} G_n \right\rangle_{\mathfrak{H}^{\otimes a_i}} \right) \left\langle \partial_{\frac{j}{n}}^{\otimes q}, h \right\rangle_{\mathfrak{H}^{\otimes q}} \right|.$$

Using (B.1), Lemma 3.5, and Lemma 2.6.a, this is bounded by

$$n^{-\theta H} \sup_{0 \le j \le \lfloor nt \rfloor} \left\| f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \right\|_{L^p(\Omega)} \prod_{i=1}^{r} \sup_j \left\| \partial_{\frac{j}{n}}^{\otimes a_i} \right\|_{\mathfrak{H}^{\otimes a_i}} \prod_{i=1}^{s} \| D^{a_i} F_n \|_{L^p(\Omega; \mathfrak{H}^{\otimes a_i})}$$
$$\times \prod_{i=s+1}^{r} \| D^{a_i} G_n \|_{L^p(\Omega; \mathfrak{H}^{\otimes a_i})} \sum_{j=0}^{\lfloor nt \rfloor -1} \left| \left\langle \partial_{\frac{j}{n}}^{\otimes q}, h \right\rangle_{\mathfrak{H}^{\otimes q}} \right| \le C n^{-(3+q)H},$$

where $p = r + 1$.

For (c), we want to consider terms of the form

$$\mathbb{E}\left|\left\langle w_n(\theta_0), \bigotimes_{i=1}^{r} D^{a_i} \Phi_n(\theta_i) \right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}\right|,$$

where $\theta_i \in \{0, 2\}$, $2 \le r \le 5 - \theta_0$, $0 \le a_i \le 4 - \theta_0$, and $a_1 + \cdots + a_r = 5 - \theta_0$. For example, the term

$$\left\langle u_n, D^3 F_n \otimes D^2 G_n \right\rangle_{\mathfrak{H}^{\otimes 3}}$$

corresponds to the case $(\theta_0, \theta_1, \theta_2) = (0, 0, 2)$, $a_1 = 3$, $a_2 = 2$. We will show that terms of this type tend to zero in $L^2(\Omega)$ as $n \to \infty$. Using the above definitions for $w_n(\theta_i)$, $\Phi_n(\theta_i)$, we have

$$\mathbb{E}\left[\left\langle w_n(\theta_0), \bigotimes_{i=1}^{r} D^{a_i} \Phi_n(\theta_i) \right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}^2\right]$$
$$= n^{-2H(\theta_0 + \cdots + \theta_r)} \mathbb{E} \sum_{p,p'=0}^{\lfloor nt \rfloor -1} \sum_{j_1,\ldots,j_r=0}^{\lfloor nt \rfloor -1} \sum_{k_1,\ldots,k_r=0}^{\lfloor nt \rfloor -1} \left\langle f^{(5)}(\widetilde{B}_{\frac{p}{n}}) \partial_{\frac{p}{n}}^{\otimes 5-\theta_0}, \bigotimes_{i=1}^{r} D^{a_i} \delta^{5-\theta_i} \left(f^{(5)}(\widetilde{B}_{\frac{j_i}{n}}) \partial_{\frac{j_i}{n}}^{\otimes 5-\theta_i} \right) \right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}$$
$$\times \left\langle f^{(5)}(\widetilde{B}_{\frac{p'}{n}}) \partial_{\frac{p'}{n}}^{\otimes 5-\theta_0}, \bigotimes_{i=i}^{r} D^{a_1} \delta^{5-\theta_i} \left(f^{(5)}(\widetilde{B}_{\frac{k_i}{n}}) \partial_{\frac{k_i}{n}}^{\otimes 5-\theta_i} \right) \right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}. \quad (27)$$

By Lemma 2.1.b,

$$D^{a_i} \delta^{5-\theta_i} \left(f^{(5)}(\widetilde{B}_{\frac{j_i}{n}}) \partial_{\frac{j_i}{n}}^{\otimes 5-\theta_i} \right)$$
$$= \sum_{\ell_i=0}^{(5-\theta_i)\wedge a_i} \ell_i! \binom{5-\theta_i}{\ell_i} \binom{a_i}{\ell_i} \delta^{5-\theta_i-\ell_i} \left(f^{(5+a_i-\ell_i)}(\widetilde{B}_{\frac{j_i}{n}}) \partial_{\frac{j_i}{n}}^{\otimes 5-\theta_i-\ell_i} \right) \partial_{\frac{j_i}{n}}^{\otimes \ell_i} \otimes \widetilde{\varepsilon}_{\frac{j_i}{n}}^{\otimes a_i - \ell_i}.$$

Applying this to each term, we can expand the inner product

$$\left\langle f^{(5)}(\widetilde{B}_{\frac{p}{n}}) \partial_{\frac{p}{n}}^{\otimes 5-\theta_0}, D^{a_1} \delta^{5-\theta_1} \left(f^{(5)}(\widetilde{B}_{\frac{j_1}{n}}) \partial_{\frac{j_1}{n}}^{\otimes 5-\theta_1} \right) \otimes \cdots \otimes D^{a_r} \delta^{5-\theta_r} \left(f^{(5)}(\widetilde{B}_{\frac{j_r}{n}}) \partial_{\frac{j_r}{n}}^{\otimes 5-\theta_r} \right) \right\rangle_{\mathfrak{H}^{\otimes 5-\theta_0}}$$

into terms of the form

$$C_\ell f^{(5)}(\widetilde{B}_{\frac{p}{n}})\delta^{b_1}\left(f^{(\lambda_1)}(\widetilde{B}_{\frac{j_1}{n}})\partial_{\frac{j_1}{n}}^{\otimes b_1}\right)\cdots\delta^{b_r}\left(f^{(\lambda_r)}(\widetilde{B}_{\frac{j_r}{n}})\partial_{\frac{j_r}{n}}^{\otimes b_r}\right)$$
$$\times\left\langle\partial_{\frac{p}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{\ell_1}\left\langle\partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{a_1-\ell_1}\cdots\left\langle\partial_{\frac{p}{n}},\partial_{\frac{j_r}{n}}\right\rangle_{\mathfrak{H}}^{\ell_r}\left\langle\partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_r}{n}}\right\rangle_{\mathfrak{H}}^{a_r-\ell_r},$$

where $C_\ell = C_\ell(\ell_1,\ldots,\ell_r)$ is an integer constant, each $b_i = 5-\theta_i-\ell_i$, and each $\lambda_i = 5+a_i-\ell_i$. It follows that (27) is a sum of terms of the form

$$C_\ell C_{\ell'} n^{-2H(\theta_1+\cdots+\theta_r)}\mathbb{E}\sum_{p,p'=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{p}{n}})f^{(5)}(\widetilde{B}_{\frac{p'}{n}})$$
$$\times\left(\sum_{j_1=0}^{\lfloor nt\rfloor-1}\delta^{b_1}\left(f^{(\lambda_1)}(\widetilde{B}_{\frac{j_1}{n}})\partial_{\frac{j_1}{n}}^{\otimes b_1}\right)\left\langle\partial_{\frac{p}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{\ell_1}\left\langle\partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{a_1-\ell_1}\right)$$
$$\times\cdots\times\left(\sum_{k_r=0}^{\lfloor nt\rfloor-1}\delta^{b'_r}\left(f^{(\lambda'_r)}(\widetilde{B}_{\frac{k_r}{n}})\partial_{\frac{k_r}{n}}^{\otimes b'_r}\right)\left\langle\partial_{\frac{p'}{n}},\partial_{\frac{k_r}{n}}\right\rangle_{\mathfrak{H}}^{\ell'_r}\left\langle\partial_{\frac{p'}{n}},\widetilde{\varepsilon}_{\frac{k_r}{n}}\right\rangle_{\mathfrak{H}}^{a_r-\ell'_r}\right).\quad(28)$$

For $0\le j_1,\ldots,j_r\le\lfloor nt\rfloor$ we have the estimate

$$\sum_{p=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{p}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{\ell_1}\left\langle\partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{a_1-\ell_1}\cdots\left\langle\partial_{\frac{p}{n}},\partial_{\frac{j_r}{n}}\right\rangle_{\mathfrak{H}}^{\ell_r}\left\langle\partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_r}{n}}\right\rangle_{\mathfrak{H}}^{a_r-\ell_r}\right|$$
$$\le\sup_{\mathcal{I}}\sum_{p=0}^{\lfloor nt\rfloor-1}\prod_{i=1}^{r}\left|\left\langle\partial_{\frac{p}{n}},\partial_{\frac{j_i}{n}}\right\rangle_{\mathfrak{H}}^{\ell_i}\left\langle\partial_{\frac{p}{n}},\widetilde{\varepsilon}_{\frac{j_i}{n}}\right\rangle_{\mathfrak{H}}^{a_i-\ell_i}\right|,$$

where $\mathcal{I}=\{0\le j_1,\ldots,j_r\le\lfloor nt\rfloor\}$. By Lemma 2.6.a and/or 2.6.c, this is bounded by $Cn^{-2H(5-\theta_0)}$ if $\ell_1+\cdots+\ell_r\ge 1$, and bounded by $Cn^{-2H(5-\theta_0-1)}=Cn^{-2H(4-\theta_0)}$ if and only if $\ell_1=\cdots=\ell_r=0$. Hence, we can write

$$\sup_{\mathcal{I},\mathcal{I}'}\sum_{p,p'=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{p}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{\ell_1}\cdots\left\langle\partial_{\frac{p'}{n}},\widetilde{\varepsilon}_{\frac{k_r}{n}}\right\rangle_{\mathfrak{H}}^{a_r-\ell'_r}\right|\le Cn^{-\Lambda H},\quad(29)$$

where $4H(4-\theta_0)\le\Lambda\le 4H(5-\theta_0)$.

It follows that terms of the form (28) can be bounded in absolute value by

$$Cn^{-2H(\theta_0+\cdots+\theta_r)}\sup_{0\le p\le\lfloor nt\rfloor}\|f^{(5)}(\widetilde{B}_{\frac{p}{n}})\|^2_{L^{4r+2}(\Omega)}\sup_{\mathcal{I},\mathcal{I}'}\sum_{p,p'=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{p}{n}},\partial_{\frac{j_1}{n}}\right\rangle_{\mathfrak{H}}^{\ell_1}\cdots\left\langle\partial_{\frac{p'}{n}},\widetilde{\varepsilon}_{\frac{k_r}{n}}\right\rangle_{\mathfrak{H}}^{a_r-\ell'_r}\right|$$
$$\times\prod_{i=1}^{r}\left\|\sum_{j_i=0}^{\lfloor nt\rfloor-1}\delta^{b_i}\left(f^{(\lambda_i)}(\widetilde{B}_{\frac{j_i}{n}})\partial_{\frac{j_i}{n}}^{\otimes b_i}\right)\right\|_{L^{2r+1}(\Omega)}\left\|\sum_{k_i=0}^{\lfloor nt\rfloor-1}\delta^{b'_i}\left(f^{(\lambda'_i)}(\widetilde{B}_{\frac{k_i}{n}})\partial_{\frac{k_i}{n}}^{\otimes b'_i}\right)\right\|_{L^{2r+1}(\Omega)}.$$

By (29) and Lemma 3.5, this is bounded by

$$C\lfloor nt\rfloor^r n^{-2H(\theta_0+\cdots+\theta_r)-\Lambda H-H(b_1+\cdots+b_r+b'_1+\cdots+b'_r)}.$$

We have $\Lambda\ge 4H(4-\theta_0)$, and

$$b_1+\cdots+b_r=5r-(\theta_1+\cdots+\theta_r)-(\ell_1+\cdots+\ell_r).$$

Since $\ell_i \le a_i$ for each i, then $\ell_1 + \cdots + \ell_r \le a_1 + \cdots + a_r = 5 - \theta_0$, it follows that the exponent

$$\begin{aligned} &2H(\theta_0 + \cdots + \theta_r) + \Lambda H + H(b_1 + \cdots + b_r + b'_1 + \cdots + b'_r) \\ &\qquad \ge 2H(\theta_0 + \cdots + \theta_r) + 4H(4 - \theta_0) + H(10r - 2(\theta_1 + \cdots + \theta_r) - 2(5 - \theta_0)) \\ &\qquad \ge 16H + 10(r-1)H \ge 10rH + 6H. \end{aligned}$$

Hence, we have an upper bound of

$$C\lfloor nt \rfloor^r n^{-10rH - 6H} \le C n^{-6H}$$

for each term of the form (28), so this term tends to zero in $L^2(\Omega)$, and we have (c). This concludes the proof of Lemma 3.6. $\square$

4.3 Proof of Lemma 3.7.

Starting with (a), Lemma 2.1.b gives

$$\begin{aligned} \mathbb{E}\left|\left\langle u_n, D^5 G_n \right\rangle_{\mathfrak{H}^{\otimes 5}}\right| &= n^{-2H} \mathbb{E}\left| \sum_{i=0}^{3} \binom{5}{i}\binom{3}{i} i! \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \partial_{\frac{j}{n}}^{\otimes 5}, \delta^{3-i}\left(f^{(10-i)}(\widetilde{B}_{\frac{k}{n}}) \partial_{\frac{k}{n}}^{\otimes 3-i}\right) \partial_{\frac{k}{n}}^{\otimes i} \otimes \widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 5-i} \right\rangle_{\mathfrak{H}^{\otimes 5}} \right| \\ &\le C n^{-2H} \sum_{i=0}^{3} \sup_{0 \le j \le \lfloor nt \rfloor} \left\| f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \right\|_{L^2(\Omega)} \sup_{0 \le k \le \lfloor nt \rfloor} \left\| \delta^{3-i}\left(f^{(10-i)}(\widetilde{B}_{\frac{k}{n}}) \partial_{\frac{k}{n}}^{\otimes 3-i}\right) \right\|_{L^2(\Omega)} \\ &\quad \times \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{i} \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{5-i} \right|. \end{aligned}$$

By moderate growth conditions and (18), we have $\left\| f^{(5)}(\widetilde{B}_{\frac{j}{n}}) \right\|_{L^2(\Omega)} \le C$ and $\left\| \delta^{3-i}\left(f^{(10-i)}(\widetilde{B}_{\frac{k}{n}}) \partial_{\frac{k}{n}}^{\otimes 3-i}\right) \right\|_{L^2(\Omega)} \le C \|\partial_{\frac{1}{n}}\|_{\mathfrak{H}}^{3-i} = C n^{-(3-i)H}$; so we have terms of the form

$$C n^{-(5-i)H} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{i} \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{5-i} \right|.$$

If $i > 0$, then (B.4) and Lemma 2.6.c give an estimate of

$$\begin{aligned} &C n^{-(5-i)H} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{i} \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{5-i} \right| \\ &\qquad \le C n^{-(15-3i)H} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{i} \right| \le C \lfloor nt \rfloor n^{-(15-3i)H} \le C n^{-2H}, \end{aligned}$$

because $i \le 3$. On the other hand, if $i = 0$, then by (B.4) and Lemma 2.6.a,

$$\begin{aligned} C n^{-5H} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{5} \right| &\le C n^{-5H} \sum_{k=0}^{\lfloor nt \rfloor - 1} \left\{ \sup_{0 \le k \le \lfloor nt \rfloor} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}} \right\rangle_{\mathfrak{H}}^{5} \right| \right\} \\ &\le C \lfloor nt \rfloor n^{-13H} \le C n^{-3H}, \end{aligned}$$

hence (a) is proved.

For (b), again using Lemma 2.1.b we can write

$$\mathbb{E}\left|\left\langle v_n, D^3F_n\right\rangle_{\mathfrak{H}^{\otimes 3}}\right| = n^{-2H}\mathbb{E}\left|\sum_{i=0}^{3}\binom{5}{i}\binom{3}{i}i!\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left\langle f^{(5)}(\widetilde{B}_{\frac{j}{n}})\partial_{\frac{j}{n}}^{\otimes 3}, \delta^{5-i}\left(f^{(8-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-i}\right)\partial_{\frac{k}{n}}^{\otimes i}\otimes\widetilde{\varepsilon}_{\frac{k}{n}}^{\otimes 3-i}\right\rangle_{\mathfrak{H}^{\otimes 3}}\right|$$
$$\leq Cn^{-2H}\sum_{i=0}^{3}\mathbb{E}\left|\sum_{j,k=0}^{\lfloor nt\rfloor-1} f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^{5-i}\left(f^{(8-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-i}\right)\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3-i}\right|.$$

We deal with three cases. First, assume $i=0$. Then we have a bound of

$$Cn^{-2H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^{5}\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right)\right|\ \left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|$$
$$\leq Cn^{-2H}\sup_{0\leq j\leq\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\leq k\leq\lfloor nt\rfloor}\left\|\delta^{5}\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right)\right\|_{L^2(\Omega)}\sup_{j,k}\left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{2}\right|$$
$$\times\sum_{k=0}^{\lfloor nt\rfloor-1}\left\{\sup_{0\leq k\leq\lfloor nt\rfloor}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\right\}\leq C\lfloor nt\rfloor n^{-11H}\leq Cn^{-H},$$

where, as above, we use the estimates $\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\leq C$ and $\left\|\delta^{5}\left(f^{(8)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5}\right)\right\|_{L^2(\Omega)}\leq Cn^{-5H}$; and

$$\sup_{j,k}\left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{2}\right|\sum_{k=0}^{\lfloor nt\rfloor-1}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|\leq C\lfloor nt\rfloor n^{-4H}$$

follows from (B.4) and Lemma 2.6.a.

The next case is for $i=1$ or $i=2$. Using similar estimates we have

$$Cn^{-2H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^{5-i}\left(f^{(8-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-i}\right)\right|\ \left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3-i}\right|$$
$$\leq Cn^{-2H}\sup_{0\leq j\leq\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^2(\Omega)}\sup_{0\leq k\leq\lfloor nt\rfloor}\left\|\delta^{5-i}\left(f^{(8-i)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 5-i}\right)\right|_{L^2(\Omega)}\sup_{j,k}\left|\left\langle \partial_{\frac{j}{n}}, \widetilde{\varepsilon}_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3-i}\right|$$
$$\times\sum_{k=0}^{\lfloor nt\rfloor-1}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{i}\right|\leq C\lfloor nt\rfloor n^{-(7-i+6)H}\leq Cn^{-H},$$

because $7-i+6\geq 11$ for $i\leq 2$.

For the case $i=3$, we will use a different estimate, and show that the term with $i=3$ vanishes in $L^2(\Omega)$. Using Lemma 2.1.d we have,

$$\mathbb{E}\left[\left(n^{-2H}\sum_{j,k=0}^{\lfloor nt\rfloor-1}f^{(5)}(\widetilde{B}_{\frac{j}{n}})\delta^{2}\left(f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 2}\right)\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\right)^{2}\right]$$
$$= n^{-4H}\sum_{j,j',k,k'=0}^{\lfloor nt\rfloor-1}\mathbb{E}\left[f^{(5)}(\widetilde{B}_{\frac{j}{n}})f^{(5)}(\widetilde{B}_{\frac{j'}{n}})\delta^{2}\left(f^{(5)}(\widetilde{B}_{\frac{k}{n}})\partial_{\frac{k}{n}}^{\otimes 2}\right)\delta^{2}\left(f^{(5)}(\widetilde{B}_{\frac{k'}{n}})\partial_{\frac{k'}{n}}^{\otimes 2}\right)\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle \partial_{\frac{j'}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3}\right]$$
$$= n^{-4H}\sum_{p=0}^{2}\binom{2}{p}^{2}p!\sum_{j,j',k,k'}\mathbb{E}\left[g(j,j')\delta^{4-2p}\left(g(k,k')\partial_{\frac{k}{n}}^{\otimes 2-p}\otimes\partial_{\frac{k'}{n}}^{\otimes 2-p}\right)\right]\left\langle \partial_{\frac{k}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{p}\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle \partial_{\frac{j'}{n}}, \partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3},$$

where $g(j,j') = f^{(5)}(\widetilde{B}_{\frac{j}{n}})f^{(5)}(\widetilde{B}_{\frac{j'}{n}})$. Then by the Malliavin duality (10), this results in a sum of three terms of the form

$$Cn^{-4H}\sum_{j,j',k,k'}\mathbb{E}\left[\left\langle D^{4-2p}g(j,j'), g(k,k')\partial_{\frac{k}{n}}^{\otimes 2-p}\otimes\partial_{\frac{k'}{n}}^{\otimes 2-p}\right\rangle_{\mathfrak{H}^{\otimes 4-2p}}\right]\left\langle\partial_{\frac{k}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{p}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle\partial_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3}, \tag{30}$$

for $p = 0, 1, 2$. When the index $p = 0$, then $\mathbb{E}\left|\left\langle D^{4-2p}g(j,j'), g(k,k')\partial_{\frac{k}{n}}^{\otimes 2-p}\otimes\partial_{\frac{k'}{n}}^{\otimes 2-p}\right\rangle_{\mathfrak{H}^{\otimes 4-2p}}\right|$ consists of terms of the form

$$\mathbb{E}\left|\left(\frac{\partial^4}{\partial x_1^a\partial x_2^b}\Psi(\widetilde{B}_{\frac{j}{n}},\widetilde{B}_{\frac{j'}{n}})\right)g(k,k')\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{a}\left\langle\widetilde{\varepsilon}_{\frac{j'}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{2-a}\left\langle\widetilde{\varepsilon}_{\frac{j}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{b}\left\langle\widetilde{\varepsilon}_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{2-b}\right|, \tag{31}$$

where $\Psi(x_1,x_2) = f^{(5)}(x_1)f^{(5)}(x_2)$ and $a + b = 4$. By moderate growth and (B.4), we see that (31) is bounded by Cn^{-8H}, and so for the case $p = 0$, (30) is bounded in absolute value by

$$\begin{aligned}Cn^{-12H}\sum_{j,j',k,k'}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle\partial_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3}\right| &= Cn^{-12H}\left(\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|\right)^2\\ &\le C\lfloor nt\rfloor^2 n^{-24H}\le Cn^{-4H}.\end{aligned}$$

By a similar estimate, when $p = 1$, then

$$\mathbb{E}\left|\left\langle D^2g(j,j'), g(k,k')\partial_{\frac{k}{n}}\otimes\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\le Cn^{-4H},$$

so that for $p = 1$, then (30) is bounded in absolute value by

$$\begin{aligned}&Cn^{-8H}\sum_{j,j',k,k'}\left|\left\langle\partial_{\frac{k}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle\partial_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|\\ &\qquad\le Cn^{-8H}\sup_{k,k'}\left|\left\langle\partial_{\frac{k}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}\right|\left(\sum_{j,k=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|\right)^2\le C\lfloor nt\rfloor^2 n^{-22H}\le Cn^{-2H}.\end{aligned}$$

Last, the term in (30) with $p = 2$ has the form

$$Cn^{-4H}\sum_{j,j',k,k'}\mathbb{E}\left[g(j,j')g(k,k')\right]\left\langle\partial_{\frac{k}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{2}\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\left\langle\partial_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3}.$$

This is bounded in absolute value by

$$Cn^{-4H}\sup_{0\le j\le\lfloor nt\rfloor}\left\|f^{(5)}(\widetilde{B}_{\frac{j}{n}})\right\|_{L^4(\Omega)}^4\sum_{k,k'=0}^{\lfloor nt\rfloor-1}\left\langle\partial_{\frac{k}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{2}\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|\sum_{j'=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j'}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|. \tag{32}$$

By Lemma 2.6.c, for every $0\le k\le\lfloor nt\rfloor$ we have

$$\sum_{j=0}^{\lfloor nt\rfloor-1}\left|\left\langle\partial_{\frac{j}{n}},\partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}^{3}\right|\le Cn^{-6H},$$

hence (32) is bounded by

$$Cn^{-16H}\sum_{k,k'=0}^{\lfloor nt\rfloor-1}\left\langle\partial_{\frac{k}{n}},\partial_{\frac{k'}{n}}\right\rangle_{\mathfrak{H}}^{2}\le C\lfloor nt\rfloor n^{-20H}\le Cn^{-10H}.$$

Lemma 3.7 is proved. □

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